EARLY UNIVERSE PHYSICS

Lecture 1

Wednesdays 14 - 16

Thursdays 14 - 16 starting 26.9 every second week

Literature

- COSMOLOGY, S. Weinberg
- PHYSICAL FOUNDATIONS OF COSMOLOGY, V. Mukhanov
- TASI Lectures: INTRODUCTION TO COSMOLOGY, M. Trodden & S. Carrol
  arxiv.org: astro-ph/0401547 (for particle physicists, interesting but more condensed/advanced)

I FACTS ABOUT THE OBSERVABLE UNIVERSE

1964: first detection of CMB. Since then observations of early universe have allowed to firmly establish the following facts:

- fine grained over distances larger than 100 Mpc ($1\ \text{Mpc} \simeq 3 \times 10^{22}\,\text{m}$), observable universe is homogeneous, isotropic & in expansion
- CMB pervades space & has average temperature corresponding to black-body radiation at $T \simeq 2.73\,\text{K}$. Is highly homogeneous $\frac{\delta T}{T} \sim 10^{-5}$
- contains baryonic matter, with roughly 1 baryon every 10^9 photons. Baryons made of $\sim 75\%$ hydrogen, $\sim 25\%$ helium + small amounts of heavier elements.

If General Relativity is good description of gravity for relevant scales in observable universe then

- Baryons contribute to $\sim 5\%$ of energy density of observable universe. The remainder is $\sim 70\%$ dark energy & $\sim 25\%$ dark matter.

Conventions : metric signature $- + + +$
Units with $\hbar = c = k_B = 1$.
$m_{Pl} = (8\pi G)^{-1/2} \simeq 10^{18}$ GeV is reduced Planck mass
with G Newton gravitational constant

II. FUNDAMENTALS OF COSMOLOGY

2.1 Geometry of spacetime

Cosmological Principle : the universe is homogeneous and isotropic

Isotropic : universe looks the same in all directions.

Smoothness of CMB is direct proof of this.

Homogeneous : universe looks the same independently of position of observer.
Distribution of galaxies & large scale structures point to its validity.

C.P constrains geometry of space.
Simplest possibility is flat space :

$$dl^2 = d\bar{x}^2$$

invariant under 3d rotations & translations.

Another possibility : 3d surface of sphere with radius a in 4d Euclidean space $(\bar{x}, u)$. Observer lives on surface of sphere

$$dl^2 = d\bar{x}^2 + du^2 \qquad \text{with} \qquad u^2 + \bar{x}^2 = a^2$$

So

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -dt^2 + a^2(t)\left[d\bar{x}^2 + K \frac{(\bar{x}\cdot d\bar{x})^2}{1-K\bar{x}^2} \right]$$

Called FRW metric

Changing variables to spherical coordinates,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Using conformal time $\tau(t) \equiv \int^t \frac{dt'}{a(t')}$

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

where $a(\tau) = a[t(\tau)]$. For $K=0$, metric in this form is equal to Minkowski metric times conformal factor $a^2(\tau)$

Note: the spherical coordinates used here are defined on space. For example, for spherical geometry, r is not the radius of the 3 sphere, but a distance defined on the surface of the 3 sphere. The radius is given by a^2

Invariant under 4d rotations.

Last possibility: hyberbolic space

$$dl^2 = d\bar{x}^2 - du^2 \qquad \text{with} \quad u^2 - \bar{x}^2 = a^2$$

where a is constant in space. Invariant under Lorentz transformations with u playing role of time.

Defining $\bar{x}' = a\bar{x}$ & $u' = au$ (drop primes)
the 3 space metrics can be written as

$$dl^2 = a^2 \left[d\bar{x}^2 + K\, du^2 \right]$$

$$K = \begin{cases} +1 & \text{spherical} \\ -1 & \text{hyperbolic} \\ 0 & \text{flat} \end{cases}$$

Spherical & hyperbolic: eliminate u using differential of $u^2 \pm \bar{x}^2 = a^2 \Rightarrow u\,du = \pm\, \bar{x} \cdot d\bar{x}$. Then

$$dl^2 = a^2 \left[d\bar{x}^2 + K \frac{(\bar{x} \cdot d\bar{x})^2}{1 - K\bar{x}^2} \right]$$

$$dl^2 > 0 \Rightarrow a^2 > 0$$

To extend to spacetime geometry, add time coordinate
Nothing prevent a to be function of time

The scale factor $a(t)$ is the only time dependent quantity in FRW metric. It contains information about expansion or contraction of space.

The rate of change of $a(t)$ is the Hubble parameter:

$$H(t) \equiv \frac{\dot{a}(t)}{a} \qquad \left(\dot{a} = \frac{\partial a}{\partial t} \right)$$

The most distant galaxies are moving away from us at speed given by the Hubble law:

$$v(t) \simeq H(t)\, d(t)$$

where d is the distance between us and the galaxy and v its velocity.

Relation discovered by E. Hubble in late 1920's, verified to high accuracy with

$$H_0 = 67.3 \pm 1.2 \ \text{km}\ \text{s}^{-1}\ \text{Mpc}^{-1}$$

where H_0 is Hubble parameter today.

Integrating the Hubble law over time, we can relate the distance d to a as

$$d(t) = a(t)\, \chi$$

where χ is constant and is called comoving coordinate

2.2 Dynamics

GR $\Rightarrow$ Einstein equations

$$G_{\mu\nu} = 8\pi G \, T_{\mu\nu}$$

geometry   energy

$G_{\mu\nu}$ is a function of the metric $g_{\mu\nu}$ only.

$T_{\mu\nu}$ is the energy-momentum tensor, contains information about energy content of the universe.
Assuming perfect fluid form:

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p\, g_{\mu\nu}$$

ρ is energy density, p is pressure and u_μ is the 4-velocity of the fluid. In rest frame $u^\mu = (1,0,0,0)$
&

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & p\, g_{ij} & \\ 0 & & & \end{pmatrix}$$

For example: scalar field ϕ in Minkowski space $(-,+,+,+) = \eta_{\mu\nu}$
with $\mathcal{L} = -\partial_\mu \phi \, \partial^\mu \phi - V(\phi)$
Applying Noether theorem to $\mathcal{L}$ for space-time transformations
$x^\mu \to x^\mu - a^\mu$ gives

$$T^{\text{scalar}}_{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \partial_\nu \phi - \mathcal{L}\, \eta_{\mu\nu}$$

Einstein equations for FRW filled with perfect fluid leads to 2 independent equations.

First Friedmann equation (F1)

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \sum_i \rho_i - \frac{k}{a^2} \,,$$

where i denotes different energy species in the universe.

F1 is a constraint equation: $\dot{a}$ is determined by energy density and curvature.

Second Friedmann equation (F2)

$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi G \sum_i p_i - \frac{k}{2a^2}$$

F1 + F2 give the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i (\rho_i + 3p_i)$$

Energy-momentum conservation gives the continuity equation

$$\dot{\rho} = -3H(\rho + p)$$

It is useful but not independent of F1 & F2.
It tells us that expansion (H) affect the local energy density ρ. Also, ρ is generally not conserved since it can be transfered between matter & spacetime geometry.

F 1 relates H to curvature parameter K and energy density ρ. If $K = 0$, universe is flat with critical energy density $\rho_c \equiv \frac{3H^2}{8\pi G}$

Taking the ratio of ρ to ρ_c gives density parameter

$$\Omega_{tot} \equiv \frac{\rho}{\rho_c}$$

which measures curvature of the universe:

$$\Omega_{tot} > 1 \Rightarrow K = +1$$

$$\Omega_{tot} < 1 \Rightarrow K = -1$$

$$\Omega_{tot} = 1 \Rightarrow K = 0$$

We can also define density parameter of single species:

$$\Omega_i \equiv \frac{\rho_i}{\rho_c}$$

Previous lecture:

Lecture 2

- FRW metric: $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$

$$k = \begin{cases} +1 & \text{positive curvature} \\ -1 & \text{negative curvature} \\ 0 & \text{flat} \end{cases}$$

- Einstein equations + perfect fluid: Friedmann equations:

F1: $H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \sum_i \rho_i - \frac{k}{a^2}$

F2: $\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi G \sum_i p_i - \frac{k}{2a^2}$

$F2 - \frac{1}{2}F1 \Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i (\rho_i + 3p_i)$ acceleration eq.

Energy-momentum conserved current:

continuity eq. $\dot{\rho} = -3H(\rho + p)$

- critical density & density parameters

$$\rho_c \equiv \frac{3H^2}{8\pi G}, \quad \Omega_{TOT} = \frac{\rho_{TOT}}{\rho_c} \begin{cases} > 1 & \Rightarrow k = +1 \\ < 1 & \Rightarrow k = -1 \\ = 1 & \Rightarrow k = 0 \end{cases}$$

- equation of state: $p = w\rho$

Non-rel matter: $p = 0$, $\rho \propto a^{-3}$

rad: $p = \frac{1}{3}\rho$, $\rho \propto a^{-4}$

cosmological cst: $p = -\rho$, $\rho = cst.$

2.3 Evolution of scale factor.

Express F1 & F2 in terms of $\tau = \int^t \frac{dt'}{a(t')}$

$$\Rightarrow \quad 3h^2 = 8\pi G \rho a^2 - 3K$$

$$h^2 + 2h' = -8\pi G p - K$$

with $h(\tau) = \frac{a'}{a}$ & $a' = \frac{\partial a}{\partial \tau}$

<u>Dust dominated universe</u>: $\rho = \rho_0 a^{-3}$ & $p = 0$

$$\cancel{3} h^2 = \frac{8\pi G \rho_0}{3a} - \cancel{3}K$$

$$2h' = -h^2 - K \quad \Rightarrow \quad h(\tau) \propto \begin{cases} \cot(\tau/2) & K=+1 \\ 2/\tau & K=0 \\ \coth(\tau/2) & K=-1 \end{cases}$$

Solve for a: $\frac{a'}{a} = h(\tau) \quad \Rightarrow \quad a(\tau) \propto \begin{cases} 1-\cos(\tau) & +1 \\ \tau^2/2 & 0 \\ \cosh(\tau)-1 & -1 \end{cases}$

Solve for t: $\frac{dt}{d\tau} = a(\tau) \Rightarrow t \propto \begin{cases} \tau - \sin(\tau) & +1 \\ \tau^3/6 & 0 \\ \sinh(\tau)-1 & -1 \end{cases}$

Simple solution for $K=0 \Rightarrow$ $\underline{a(t) \propto t^{2/3}}$

Radiation dominated universe

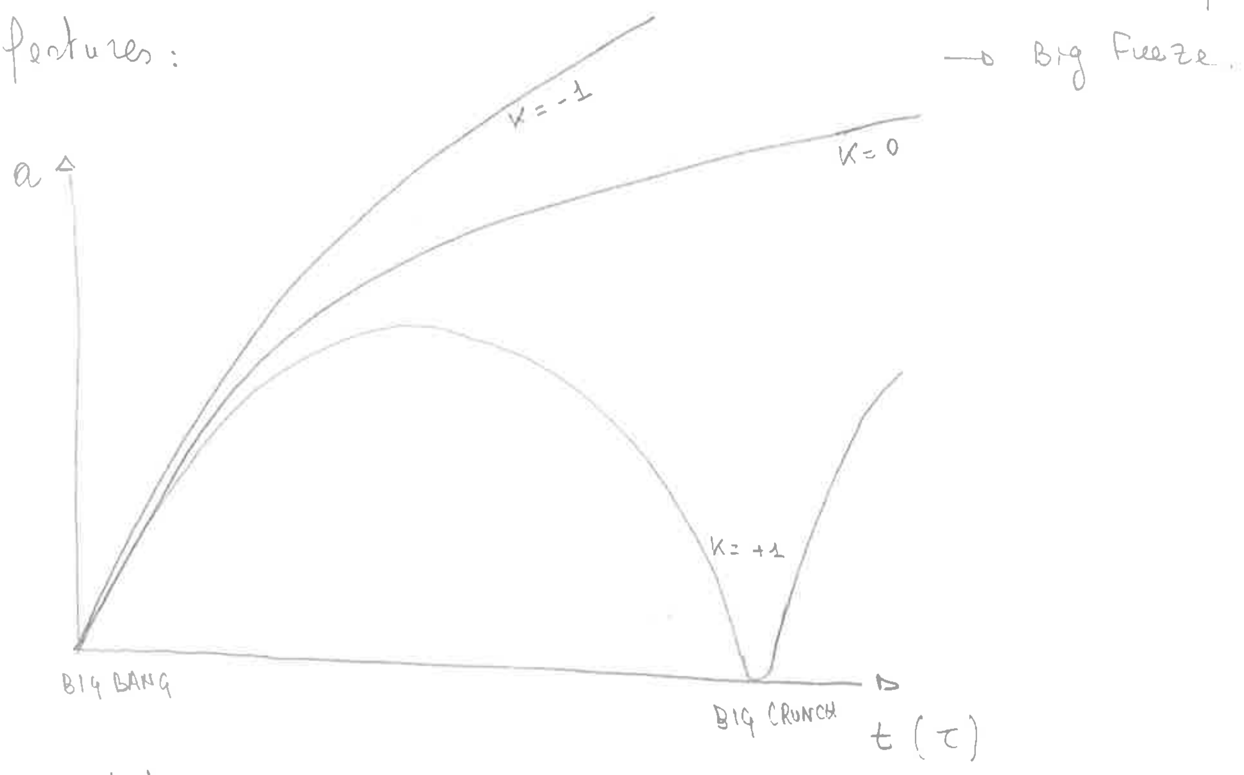
Same, but $w = \frac{1}{3}$ Find

$$h(\tau) \propto \begin{cases} \cot(\tau) & +1 \\ 1/\tau & 0 \\ \coth(\tau) & -1 \end{cases} \qquad a(\tau) \propto \begin{cases} \sin(\tau) & +1 \\ \tau & 0 \\ \sinh(\tau) & -1 \end{cases}$$

$$t \propto \begin{cases} 1 - \cos(\tau) & +1 \\ \tau^2/2 & 0 \\ \cosh(\tau) - 1 & -1 \end{cases}$$

for $k = 0 \Rightarrow a(t) \propto t^{1/2}$

Both dust & radiation filled universes have same qualitative features:



⇒ dust & radiation: positive curvature ⇒ cyclic universe with Big Crunch

negative or no curvature: expand forever after the Big Bang & end in a Big Freeze

Universe dominated by a cosmological constant

$w = -1$ & $\rho = -p$, $\rho = cst$

F1 & F2 can be solved in terms of cosmic time t.

Acceleration eq: $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(-2\rho) = \frac{8}{3}\pi G\rho = \frac{\Lambda}{3} > 0$$

Find $$a(t) \propto \begin{cases} \cosh\left(\sqrt{\frac{\Lambda}{3}}\, t\right) & +1 \\ \exp\left(\sqrt{\frac{\Lambda}{3}}\, t\right) & 0 \\ \sinh\left(\sqrt{\frac{\Lambda}{3}}\, t\right) & -1 \end{cases}$$

Independently of curvature, the 3 solutions expand forever, with accelerated expansion

In fact, they all describe the same spacetime, but in different coordinates. This is de Sitter spacetime.

2.4 Causality & horizons

Observer: receives information from past
Emitter: sends information to the future

Boundary in the past from which info can reach an observer is particle horizon

Boundary in future to which an emitter can send info is event horizon

Particle horizon : using comoving spherical coordinates (r, θ, φ), consider emitter at $r=0$. Set $d\theta = d\varphi = 0$ & $ds^2 = -d\tau^2 + \frac{dr^2}{1-kr^2}$

Light rays: $ds^2 = 0$ & $d\tau = \frac{dr}{\sqrt{1-kr^2}}$

$$\Rightarrow \underbrace{\tau_0}_{\text{observer}} - \underbrace{\tau_e}_{\text{emitter}} \equiv \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}}$$

If τ_e has a minimal value, as in FRW originating from a Big Bang, then there exists a maximal comoving distance information can travel.

In physical units:

$$d_{ph} = a(t_0) \int_0^{\bar{r}_e} \frac{dr}{\sqrt{1-kr^2}} = a(t_0) \int_{\bar{t}_e}^{t_0} \frac{dt}{a(t)}$$

Event horizon : If τ_0 has maximal value $\bar{\tau}_0$, then there exists maximal distance to which emitter can send signals

$$d_{eh} = a(t_e) \int_{\bar{r}_0}^{r_e} \frac{dr}{\sqrt{1-kr^2}} = a(t_e) \int_{t_e}^{\bar{t}_0} \frac{dt}{a(t)}$$

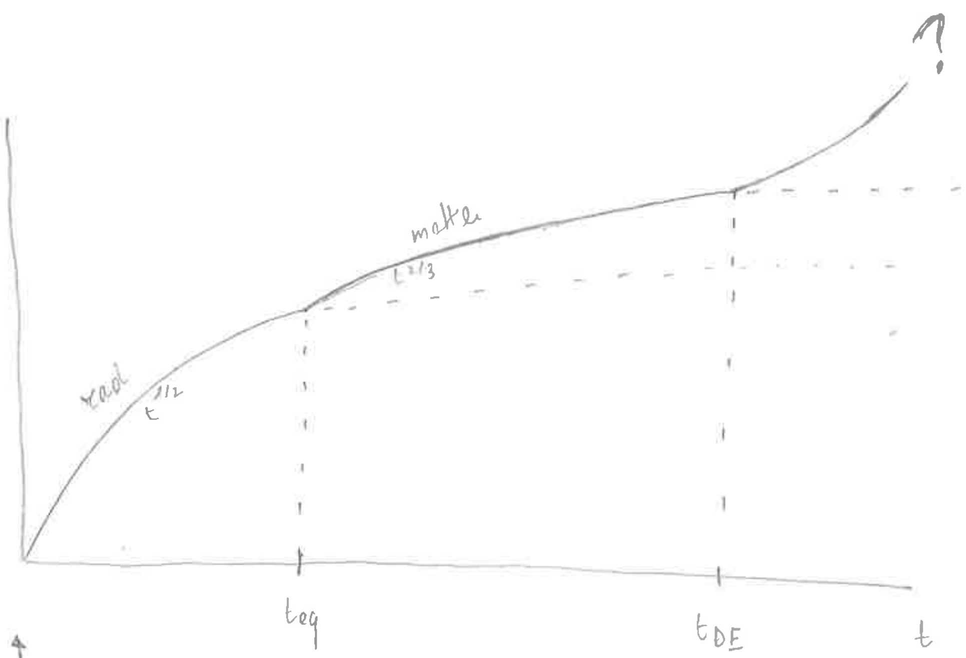
Hubble Scale : $d_H = \frac{1}{H}$, $\frac{1}{aH}$ in comoving units.

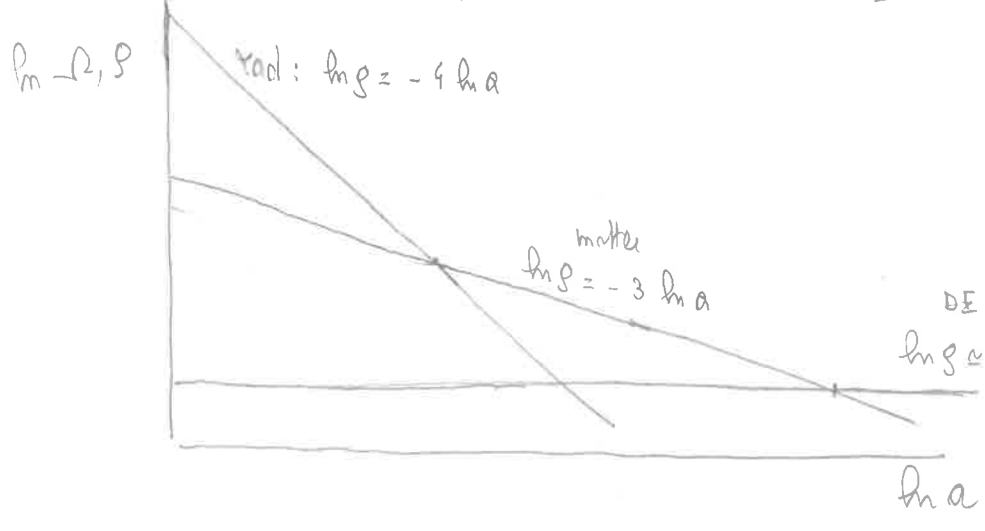
It gives size of a sphere beyond which objects move away at speed greater than light for observer at centre of sphere.

III Observed Universe

Composition.

$\Omega_{\Lambda CDM} \simeq 1$

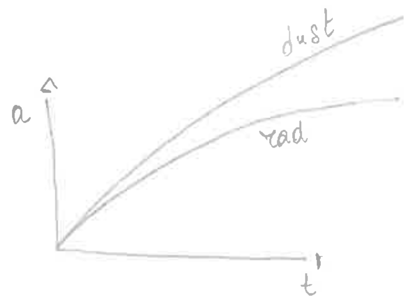




Previous lecture

- evolution of scale factor ($K = 0$)

  - dust dominated $\quad a \propto t^{2/3}$
  - radiation dominated $\quad a \propto t^{1/2}$



  - cosmological constant dominated $\quad a \propto e^{\sqrt{\frac{\Lambda}{3}}\, t}$

- causality & horizons

Particle horizon : Boundary in the past from which can receive info

$$\tau_0 - \bar{\tau}_e \equiv \int_{\bar{t}_e}^{t_0} \frac{dt}{a(t)} \quad \& \quad d_{ph} = a(t_0) \int_{\bar{t}_e}^{t_0} \frac{dt}{a(t)}$$

$\bar{\tau}_e$ ↓ minimal value

Event horizon : Boundary in the future to which can send information

$$\bar{\tau}_0 - \tau_e \equiv \int_{t_e}^{\bar{t}_0} \frac{dt}{a(t)} \quad \& \quad d_{eh} = a(t_e) \int_{t_e}^{\bar{t}_0} \frac{dt}{a(t)}$$

$\bar{\tau}_0$ ↓ maximal value

Hubble "horizon" : $d_H = \dfrac{1}{H}$, objects at $d > d_H$ move away at speed $v > 1$. No problem with relativity since info from them does not reach us.

- ΛCDM , $\Omega \simeq 1$

$$\Omega = \overbrace{(\Omega_b + \Omega_{DM})}^{\Omega_M}\left(\frac{a}{a_0}\right)^{-3} + \overbrace{(\Omega_{CMB} + \Omega_{CYB})}^{\Omega_{RAD}}\left(\frac{a}{a_0}\right)^{-4} + \Omega_{DE}$$

$\Omega_M \simeq 0.03$

$\Omega_{DE} \simeq 0.07$

$\Omega_{RAD} \simeq 0 \quad (10^{-5})$

More on observations & measurement of those quantities later in course

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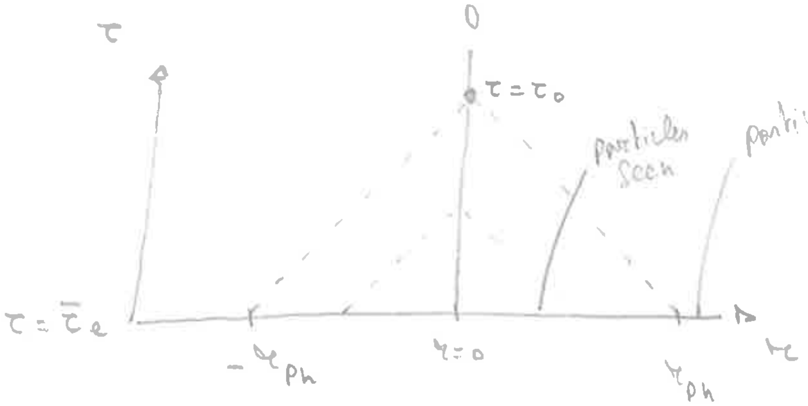
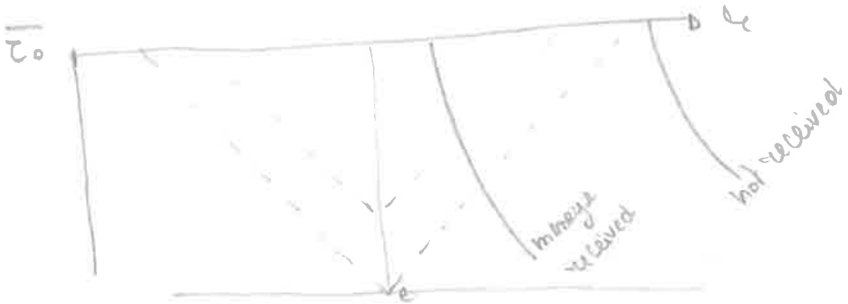
Particle horizons



Event horizon



Lecture 3

3.2 Thermal history

10^{-43} - ? s (10^{19} GeV and above) : QG regime
↳ Planck scale

10^{-14} - 10^{-43} s (10 TeV - 10^{19} GeV) : composition unknown
Extension of Standard Model describe particles & interactions
QFT + GR are good descriptions of nature
- At $\sim 10^{16}$ GeV strong & electroweak interactions unification may take place → GUT
- Baryon asymmetry originates
- Inflation

10^{-10} - 10^{-14} s (100 GeV - 10 TeV) : energies explored in accelerators
Above 100 GeV electroweak symmetry is restored

10^{-5} s (200 MeV) : quarks & gluons form baryons & mesons

0.2 s (1-2 MeV) : Neutrinos decouple → universe is transparent to them : CνB !
Ratio of neutrons to protons freezes out

1 s (0.5 MeV) : Electron - positron pairs annihilate
$e^+ + e^- \to \gamma$ raising the temperature of CMB. 1 e^- per 10^9 photons survives annihilation

200 - 300 s (0.05 MeV) : primordial nucleosynthesis : nuclear reactions become efficient & light elements are formed : H, He & Li

10^{11} s (1 eV) : Matter - radiation equality

$10^{12} - 10^{13}$ s & : CMB

$10^{16} - 10^{17}$ s : Galaxies, clusters of galaxies form from initial fluctuations as a result of gravitational interactions

3.3 Thermodynamic processes

Early universe: hot & dense $\Rightarrow$ thermodynamics
As long as microscopic processes (star formation, clusters) are negligible
expansion can be treated as adiabatic $\rightarrow$ reversible

3.3.1 Equilibrium

If interactions are effective, particles are in local thermal and chemical equilibrium

Thermal eq: exchange of energy
Chemical eq: relative numbers are conserved
This is state of maximal entropy.

Expansion affects interactions: consider mean collision time of set of particles:

$$\bar{t}_c \simeq \frac{1}{\sigma n v}$$

σ cross-section
n number density
v relative velocity

If collision time small compared to Hubble time

$$\bar{t}_c \ll \frac{1}{H}$$

interactions are frequent & expansion can be neglected
However, expansion $\rightarrow$ n decreases $\rightarrow$ $\bar{t}_c$ decrease
Eventually $\bar{t}_c \simeq \frac{1}{H}$ and interaction freezes out.
This occurred at about 1-2 MeV for neutrinos &
0.5 MeV for photons

Number of particles, energy density & pressure

Phase space $(\bar{x}, \bar{p})$, both in physical coordinates.

# of particles: count phase space elements × occupation numbers

Phase space element: $\dfrac{d^3\bar{x}\; d^3\bar{p}}{(2\pi)^3}$

Occupation numbers: Fermi-Dirac (fermions) or Bose-Einstein (Bosons) distributions:

$$f(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1} \qquad \begin{array}{l} + \text{ fermions} \\ - \text{ bosons} \end{array}$$

$E(p) = \sqrt{m^2 + p^2}$ is energy & μ is chemical potential, negligible for equilibrium processes.

$f(p)$ is homogeneous & isotropic $\Rightarrow \int d^3\bar{x} = V$.

Number density:

$$n_i \equiv \frac{N_i}{V} = g_i \int \frac{d^3\bar{p}}{(2\pi)^3} f_i(p)$$

$$= \frac{g_i}{2\pi^2} \int_0^\infty f_i(p)\, p^2\, dp$$

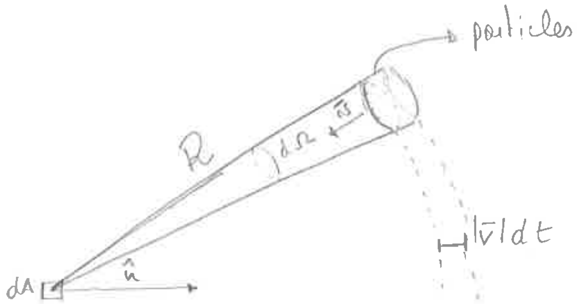
where g_i is number of internal degrees of freedom (spin)

Energy density: $\int dn_i(p)\, E(p)$

$$\Rightarrow \rho_i = \frac{g_i}{2\pi^2} \int_0^\infty f_i(p)\, E(p)\, p^2\, dp$$

Finally, pressure = force / area = $\dfrac{\text{momentum}}{\text{time interval}} \times \dfrac{1}{\text{Area}}$

Area element dA with normal vector $\hat{n}$.



Particles with speed $\bar{v}$ hitting dA between times t & $t+dt$, coming from within $d\Omega$ at distance $R = |v|t$ at $t=0$.

Total number of particles inside the shell:

$$dN = dn\, R^2\, d\Omega\, |v|\, dt$$

Only particles hitting dA contribute to pressure:

$$dN_{dA} = \frac{(\bar{v}\cdot\hat{n})}{|\bar{v}|} \frac{dA}{4\pi R^2}\, dN = \frac{(\bar{v}\cdot\hat{n})}{4\pi}\, dn\, dt\, d\Omega\, dA$$

Assume elastic scattering: each particle hitting dA transfers $2(\bar{p}\cdot\hat{n})$ momentum, with $|p| = E|\bar{v}|$

Then pressure on dA is:

$$dP = \int \frac{2(\bar{p}\cdot\hat{n})\, dN_{dA}}{dA\, dt} = \frac{p^2}{2\pi E}\, dn \int \cos^2\theta\, d\Omega$$

$$= \frac{p^2}{3E}\, dn$$

For a single species, $P_i = \dfrac{g_i}{2\pi^2}\displaystyle\int_0^\infty \frac{f_i(p)\, p^4\, dp}{3\,E(p)}$

Entropy

If we treat universe as closed thermodynamic system entropy is useful.

First law of thermodynamics :

$$dE = T dS - P dV$$

$E(T, V)$ & $S(T, V)$.

$$\Rightarrow \frac{\partial E}{\partial V} dV + \frac{\partial E}{\partial T} dT = \left(\frac{\partial S}{\partial V} dV + \frac{\partial S}{\partial T} dT \right) T - P dV$$

E & S are extensive : $\frac{\partial S}{\partial V} = \frac{S}{V}$ & $\frac{\partial E}{\partial V} = \frac{E}{V}$

For $dT = 0$, find

$$S = \frac{E + PV}{T}$$

Using previous results, entropy density is

$$s_i \equiv \frac{S_i}{V} = \frac{g_i}{2\pi T} \int_0^\infty f_i(p)\, p^2\, dp \left(E + \frac{p^2}{3E} \right)$$

The solutions of the integrals for ρ_i, P_i, n_i, s_i if $\mu = 0$ are given by Riemann Zeta function $\zeta(z)$

Relativistic limit : $E(p) = p$

Non - relativistic limit : $E(p) = m$

$\Rightarrow$ Relativistic :

	bosons	fermions	
n	$g \frac{\zeta(3)}{\pi^2} T^3$	$g \frac{3}{4} \frac{\zeta(3)}{\pi^2} T^3$	
ρ	$g \frac{\pi^2}{30} T^4$	$g \frac{7}{8} \frac{\pi^2}{30} T^4$	
P	$\rho / 3$	$\rho / 3$	$\left(w = \frac{1}{3} \right)$
s	$3.602\, n$	$4.202\, n$	

Non - Relativistic

n	$g\, e^{-\frac{m}{T}} \left(\frac{Tm}{2\pi} \right)^{3/2}$	
ρ	$n m$	
P	$n T \ll \rho$	$(w \simeq 0)$
s	$\left(\frac{m}{T} - \frac{5}{2} \right) n$	

3.3.2 Non-equilibrium

expansion: interactions less and less efficient in maintaining thermal (creation / annihilation of particles) and chemical ($\Sigma \mu_i = 0$) equilibrium.

Progressively, particles fall out of equilibrium:

Number densities become constant and interactions with other particles stop (decoupling).

This is freeze-out.

It happened to
- gravitational waves (presumably) ($\sim 10^{19}$ GeV)
- neutrinos ~ 1 MeV (0.2 s)
- photons $\sim$ 300.000 years
- dark matter (presumably) (before nucleosynthesis)

They all carry information about state of the universe at their decoupling time

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Assume interactions are dominated by 2-body contributions

Naively, decoupling occurs when

$$\frac{\Gamma}{H} \simeq 1 \quad , \quad \Gamma \text{ interaction rate.}$$

Distribution function out of equilibrium is not as simple as FD or BE. ⑤

To find it, need to solve Boltzmann equation

$$\hat{L}[f] = C[f]$$

Liouville operator describes phase space evolution of distribution f

Collision operator contains information about interactions.

Non-relativistic form of $\hat{L}$

$$\hat{L}_{NR} = \frac{\partial}{\partial t} + \bar{v} \cdot \nabla_{\bar{x}} + \frac{\bar{F}}{m} \cdot \nabla_{\bar{v}}$$

Relativistic generalization:

$$\hat{L} = p^{\alpha} \frac{\partial}{\partial x^{\alpha}} - \Gamma^{\alpha}_{\beta\gamma} p^{\beta} p^{\gamma} \frac{\partial}{\partial p^{\alpha}}$$

Christoffel symbols: contain information about "straight lines" in curved space.

For FRW:

$$\hat{L}[f] = E \frac{\partial f}{\partial t} - H |\bar{p}|^2 \frac{\partial f}{\partial E}$$

or in terms of number density: $\displaystyle \frac{1}{a^3} \frac{d(na^3)}{dt} = \frac{g}{(2\pi)^3} \int d^3\bar{p} \, \frac{C[f(\bar{p})]}{E}$

3.4 Particle annihilation

example : system containing 2 species of particle + antiparticle, both stable (no decay)

Only possible interactions are $p/\bar{p}$ annihilation

(e.g. $e^+ e^-$ and $\nu, \bar{\nu}$ mediated by Z boson)

$$\psi \bar{\psi} \longleftrightarrow \chi \bar{\chi}$$

$C[f]$ is calculated from QFT, and

$$\frac{1}{a^3} \frac{d(n_\psi a)}{dt} = \int \frac{d^3\bar{p}_\psi \, d^3\bar{p}_{\bar{\psi}} \, d^3\bar{p}_\chi \, d^3\bar{p}_{\bar{\chi}}}{(2\pi)^{12} E_\psi E_{\bar{\psi}} E_\chi E_{\bar{\chi}}}$$

$$\times (2\pi)^4 \delta^3(\bar{p}_\psi + \bar{p}_{\bar{\psi}} - \bar{p}_\chi - \bar{p}_{\bar{\chi}}) \delta(E_\psi + E_{\bar{\psi}} - E_\chi - E_{\bar{\chi}})$$

$$\times \left(f_\chi f_{\bar{\chi}} [1 \pm f_\psi][1 \pm f_{\bar{\psi}}] - f_\psi f_{\bar{\psi}} [1 \pm f_\chi][1 \pm f_{\bar{\chi}}] \right)$$

$$|\mathcal{M}|^2$$

We set $g = 1$. $\mathcal{M}$ is scattering amplitude from QFT.

$\delta(\Sigma \bar{p}) \to$ momentum conservation, $\delta(\Sigma E)$ energy conservation

$1 \pm f$: + boson , − fermions. If particle exists interaction producing more is more likely for bosons than fermions

Furthermore: production of ψ proportional to # of χ & $\bar{\chi}$
annihilation of ψ " " " " ψ & $\bar{\psi}$

The integrals should be over 4-momenta, but using $E^2 = p^2 + m^2$, we have

$$\int d^3\bar{p} \int dE \, \delta(E^2 - p^2 - m^2) = \int d^3\bar{p} \, dE \frac{\delta\left(E - \sqrt{p^2+m^2}\right)}{2E}$$

$$= \int \frac{d^3\bar{p}}{2\sqrt{p^2+m^2}}$$

Interested in regime out of chemical equilibrium

$$E - \mu >> T$$

As temperature falls Γ is too low to allow thermalization of particles - antiparticles $\rightarrow$ can neglect quantum nature of particles & BE, FD $\rightarrow$ Boltzmann distribution

$$f = \frac{1}{e^{(E-\mu)/T} \pm 1} \rightarrow \exp\left[-\frac{E-\mu}{T}\right]$$

In this limit

$$f_\chi f_{\bar{\chi}} [1 \pm f_\psi][1 \pm f_{\bar{\psi}}] - f_\psi f_{\bar{\psi}} [1 \pm f_\chi][1 \pm f_{\bar{\chi}}]$$

$$\rightarrow e^{-(E_\psi + E_{\bar{\psi}})/T} \left(e^{(\mu_\chi + \mu_{\bar{\chi}})/T} - e^{(\mu_\psi + \mu_{\bar{\psi}})/T} \right)$$

where we used energy conservation $E_\psi + E_{\bar{\psi}} = E_\chi + E_{\bar{\chi}}$

3.3.2 Non - equilibrium

equilibrium: - thermal: interaction rates high and exchange of energy is efficient.

- chemical: reactions are spontaneous: can go in both directions & relative number densities are conserved.

Expansion: rate of interactions decreases.

When $\Gamma \simeq H$, interaction freezes out.

As freeze out is approached, system falls out of equilibrium

$\longrightarrow$ distribution function given by Boltzmann Equation

$$\hat{L}[f] = C[f]$$

For FRW:

$$\frac{1}{a^3}\frac{d(na^3)}{dt} = \int \frac{g\, d^3\bar{p}}{(2\pi)^3} \frac{C[f(\bar{p})]}{E(\bar{p})}$$

3.4 Particle annihilation

System with 2 species $\psi, \bar{\psi}$ & $\chi, \bar{\chi}$

$$\psi \bar{\psi} \longleftrightarrow \chi \bar{\chi}$$

Out of chemical & thermal equilibrium: $E - \mu \gg T$

$\Longrightarrow$ BE & FD distributions reduce to Boltzmann distribution

$$f \longrightarrow \exp\left[-\frac{E-\mu}{T}\right]$$

number density simplifies

$$n_i = g_i \, e^{\mu_i/T} \int \frac{d^3\bar{p}}{(2\pi)^3} \, e^{-E_i/T} \equiv e^{\mu_i/T} \, n_i^0$$

where n_i^0 is number density of chemical equilibrium ($\mu = 0$).

Then

$$f_\chi f_{\bar\chi} [1 \pm f_\psi][1 \pm f_{\bar\psi}] - f_\psi f_{\bar\psi} [1 \pm f_\chi][1 \pm f_{\bar\chi}]$$

$$\longrightarrow e^{-(E_\psi + E_{\bar\psi})/T} \left(e^{(\mu_\chi + \mu_{\bar\chi})/T} - e^{(\mu_\psi + \mu_{\bar\psi})/T} \right)$$

$$\longrightarrow e^{-(E_\psi + E_{\bar\psi})/T} \left(\frac{n_\chi n_{\bar\chi}}{n_\chi^0 n_{\bar\chi}^0} - \frac{n_\psi n_{\bar\psi}}{n_\psi^0 n_{\bar\psi}^0} \right).$$

Define total thermally averaged cross-section assuming scattering is istantaneous

$$\langle \sigma v \rangle \equiv \frac{1}{n_\psi^0 n_{\bar\psi}^0} \int \frac{d^3\bar{p}_\psi \, d^3\bar{p}_{\bar\psi} \, d^3\bar{p}_\chi \, d^3\bar{p}_{\bar\chi}}{(2\pi)^{12} \, E_\psi E_{\bar\psi} E_\chi E_{\bar\chi}}$$

$$\times (2\pi)^4 \, \delta^3(\Sigma \bar{p}) \, \delta(\Sigma E) \, e^{-(E_\psi + E_{\bar\psi})/T} \, |\mathcal{M}|^2$$

Boltzmann equation becomes

$$\frac{1}{a^3} \frac{d(n_\psi a^3)}{dt} = n_\psi^0 n_{\bar\psi}^0 \langle \sigma v \rangle \left(\frac{n_\chi n_{\bar\chi}}{n_\chi^0 n_{\bar\chi}^0} - \frac{n_\psi n_{\bar\psi}}{n_\psi^0 n_{\bar\psi}^0} \right)$$

$\Gamma \equiv n_{\bar\psi} \langle \sigma v \rangle$ is interaction rate.

Left-hand side goes as $n_\psi H$ for $n_\psi = $ cst & we recover intuitive factor $\frac{\Gamma}{H}$.

Statistical equilibrium : $\Gamma \gg H$ & left hand side small $\Rightarrow$

$$\frac{n_\chi n_{\bar\chi}}{n^0_\chi n^0_{\bar\chi}} = \frac{n_\psi n_{\bar\psi}}{n^0_\psi n^0_{\bar\psi}}$$

This is Saha equation

3.5 Nucleosynthesis

Universe at 0.5 MeV made of

photons γ, e^+, e^-, neutrons (n), protons (p)

3 neutrinos & anti-neutrinos (ν_e, ν_μ, ν_τ)

Some mesons (quark - anti-quarks), μ & τ leptons

& heavy baryons but not significant densities.

Dark matter also present, decoupled.

ν decoupled at ~ 1 MeV.

Baryons & anti-baryons annihilated with baryon asymmetry

$$\frac{n_b - n_{\bar b}}{n_\gamma} \sim 10^{-10}$$

generated via baryogenesis at higher energies

Above 1 MeV interactions rate is high & protons, neutrons are in thermal & chemical equilibrium

At ~ 0.1 MeV interactions become inefficient and neutron abundance becomes fixed $\rightarrow$ initial condition for nucleosynthesis

Nucleosynthesis $\to$ H, He, D, T
deuterium ^2H 1 proton + 1 neutron
tritium $\to$ ^3H 1 proton + 2 neutrons

Some heavier elements (negligible)

3.5.1 Neutron abundance

Weak interaction processes

$$n \longleftrightarrow p + e^- + \bar{\nu}_e \quad (1)$$

$$p + e^- \longleftrightarrow n + \nu_e \quad (2)$$

$$p + \bar{\nu}_e \longleftrightarrow n + e^+ \quad (3)$$

Non-relativistic limit: $E_i \simeq m_i$, proton-neutron at equilibrium $\Rightarrow$ $\dfrac{n_p^0}{n_n^0} \simeq e^{(m_n - m_p)/T}$, $m_n - m_p = 1.29\,\text{MeV}$

In non-expanding universe $\dfrac{n_p^0}{n_n^0}$ grows until $n_n^0 = 0$.

Expansion $\to$ weak interaction decouples before, fixing $\dfrac{n_p}{n_n}$

Energies above 1 MeV $\to$ protons & neutrons complete equilibrium
" below " $\to$ solve Boltzmann equation

(2) & (3) are annihilation processes

For (2), we have

$$\frac{1}{a^3}\frac{d(n_n a^3)}{dt} = n_n^0 \, n_{\nu_e}^0 \langle \sigma v \rangle \left(\frac{n_p n_{e^-}}{n_p^0 n_{e^-}^0} - \frac{n_n n_{\nu_e}}{n_n^0 n_{\nu_e}^0} \right)$$

Leptons are still in equilibrium : $n_{e^-} = n_{e^-}^0$ & $n_{\nu_e} = n_{\nu_e}^0$

$$\frac{1}{a^3}\frac{d(n_n a^3)}{dt} = n_{\nu_e}^0 \langle \sigma v \rangle \left(\frac{n_p n_n^0}{n_p^0} - n_n \right)$$

Change variables $X_n = \dfrac{n_n}{n_n + n_p}$ & $\Gamma_{np} = n_{\nu_e}^0 \langle \sigma v \rangle$

$$\frac{dX_n}{dt} + 3HX_n = \Gamma_{np}\left[(1 - X_n)\, e^{-(m_n - m_p)/T} - X_n \right]$$

T is also function of time : radiation domination

$$\rho = g_* \frac{\pi^2}{30} T^4 \quad \& \quad \rho \propto a^{-4} \implies T \propto a^{-1}$$

$$g_* = \sum_{\substack{i \\ \text{bosons}}} g_i + \frac{7}{8} \sum_{\substack{j \\ \text{fermions}}} g_j$$

At ~0.5 MeV : photons, $g = 2$; ν & $\bar{\nu}$ $g = 3 \times 2$,

e^+ & e^- $g = 2 + 2$

$$\implies g_* = 10.75$$

Use $x = \dfrac{m_n - m_p}{T}$ as time variable

$$\frac{dx}{dt} = -\frac{x}{T}\frac{dT}{dt} = xH = x\sqrt{\frac{8\pi G \rho}{3}}$$

We have

$$\frac{dX_n}{dx} + \frac{3}{x} X_n = \sqrt{\frac{45}{4\pi G g_*}} \frac{x\, \Gamma_{np}(x)}{(m_n - m_p)^2} \left[e^{-x} - X_n (1+e^{-x}) \right]$$

$$\sim \frac{\Gamma_{np}}{x H} \left[e^{-x} - X_n \left(1 + e^{-x} \right) \right]$$

Γ_{np} depends on T :

$$\Gamma_{np} = \frac{255}{\tau_n x^5} \left(12 + 6x + x^2 \right)$$

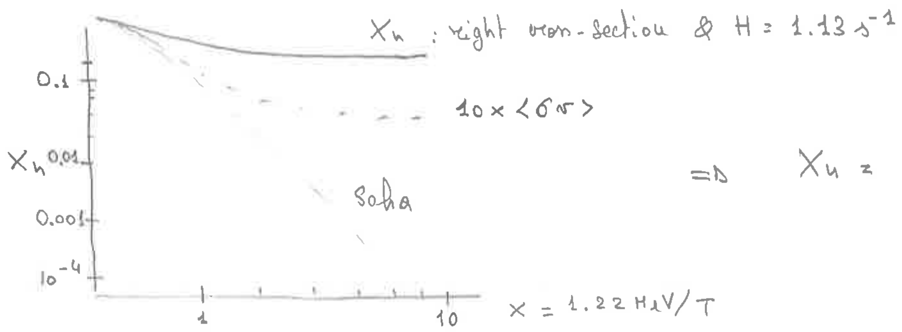
$\tau_n = 886{,}7\ s$ neutron half-life.

At $x = 1$, $T = m_n - m_p$, $n \longleftrightarrow p$ conversion rate is $\Gamma_{np} \simeq 5.5\ s^{-1}$ & $H \simeq 1.1\ s^{-1}$, interactions still effective.

(1) $n \longleftrightarrow p + e^- + \bar{\nu}$ contributes with suppression factor e^{-t/τ_n}. It becomes important at ~ 0.1 MeV

At such time e^+, e^- have annihilated (e^- survive) they do not contribute to energy density & $g_* = 3.36$

To find X_n, equation has to be solved numerically.



$\Rightarrow X_n \simeq 0.11$

3.5.2 Light elements formation

D & He formed through electromagnetic interactions

$$p + n \longleftrightarrow D + \gamma \qquad (1)$$

$$D + D \longleftrightarrow n + {}^3He \qquad (2)$$

$$^3He + D \longleftrightarrow p + {}^4He \qquad (3)$$

interactions involving 3 or 4 nucleons can produce He directly, but much less likely $\rightarrow$ Deuterium bottleneck.

^{12}C could be produced via $^4He + {}^4He + {}^4He \longleftrightarrow {}^{12}C$ but densities are too low. Heavier elements are produced in stars & supernovae.

$T > 1$ MeV nuclei are unstable due to interactions with high energy photons.

Remember $\quad \eta_b = \frac{n_b}{n_\gamma} \sim 8 \times 10^{-10} \, \Omega_b h$

Apply Saha equation to (1)

$$\frac{n_D \, n_\gamma}{n^0_D \, n^0_\gamma} = \frac{n_n \, n_p}{n^0_n \, n^0_p}$$

Approximate $n_\gamma = n^0_\gamma$

$$\frac{n_D}{n_n n_p} = \frac{n^0_D}{n^0_n \, n^0_p}$$

Using non-relativistic result for equilibrium densities

$$\frac{n_D}{n_n n_p} = \frac{3}{4} \left(\frac{2\pi \, m_D}{m_n \, m_p T} \right)^{3/2} e^{(m_n + m_p - m_D)/T}$$

with $g_D = 3$ & $g_p = g_n = 2$

n_p & n_n proportional to n_b. We can rewrite equation as

$$\frac{n_D}{n_b} \sim \eta_b \left(\frac{T}{m_p} \right)^{3/2} e^{B_D/T}$$

where $B_D \equiv m_p + m_n - m_D$ (Deuterium binding energy)
$= 2.22\,\text{MeV}$

$\eta_b = \frac{n_b}{n_\gamma}$, $n_n n_p \sim n_b^2$, $n_\gamma \propto T^3$, $\frac{m_D}{m_n m_p} \sim \frac{1}{m_p}$

η_b small $\Rightarrow$ $\frac{n_D}{n_b}$ also small until $B \gg T$

Assuming Deuterium forms instantaneously at T_{nuc} & $n_D \sim n_b$

$$\ln \eta_b + \frac{3}{2} \ln T_{nuc} - \frac{3}{2} \ln m_p \sim - \frac{B_D}{T_{nuc}}$$

$\Rightarrow$ $T_{nuc} \sim 0.07$ MeV

Binding energy of helium larger than B_D & most D combines almost immediately to form helium via (2) & (3)

Assume all neutrons end up in ^4He $\to$ contains 2 neutrons so final abundance of ^4He is half neutron abundance at T_{nuc}

Then $X_4 = \frac{4 n_4}{n_b} = 2 X_n(T_{nuc})$,

$X_n(T_{nuc}) \simeq 0.11$ $\Rightarrow$ $X_4 \simeq 0.22$ compared to ~ 0.25 observed in astrophysical experiments.

Some D remains because freeze-out happens before saturation.

Nucleosynthesis sensitive to n_b: less n_b, slower conversion

Lecture 5

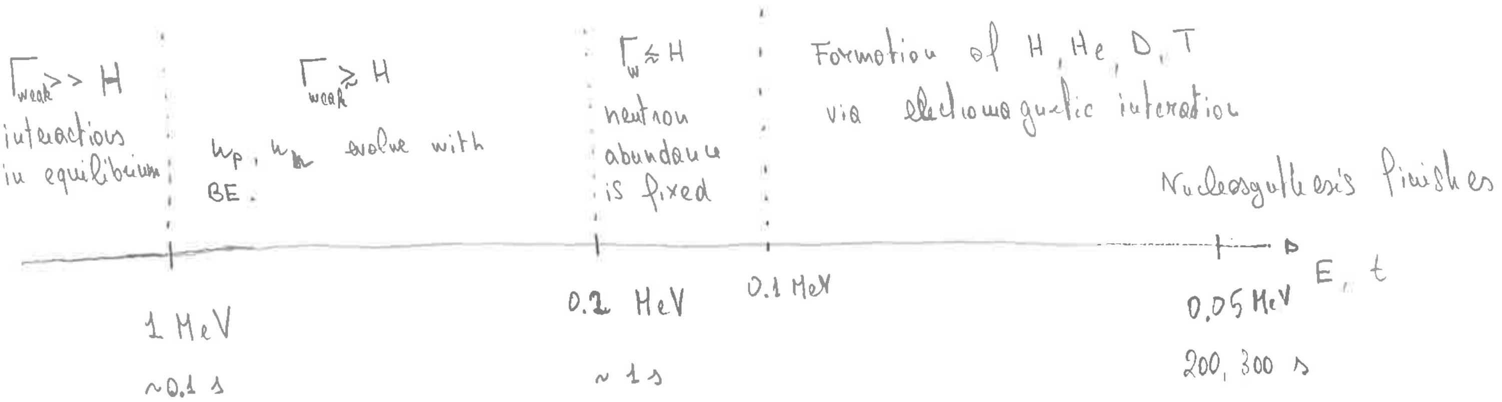
Nucleosynthesis:

At 0.5 MeV : universe radiation dominated

species present are γ, e^+, e^-, neutrons, protons
3 ν & $\bar{\nu}$, few mesons $(q-\bar{q})$, μ, τ leptons & heavy baryons.

Baryons & anti-baryons annihilated leaving excess of baryons

Timeline of nucleosynthesis



brief overview of calculation
ingredients: Boltzmann equation & QFT

Neutron abundance

Fixed when weak interaction freezes out.

$$n \longrightarrow p + e^- + \bar{\nu}_e \qquad (1)$$

is decay of neutron, giving suppression factor $n_p \propto e^{-t/\tau_n}$ where $\tau_n = 886.7\,s$ is half-life of neutron

$$p + e^- \longrightarrow n + \nu_e \qquad (2)$$

$$p + \bar{\nu}_e \longrightarrow n + e^+ \qquad (3)$$

Are annihilation processes.

Neglecting expansion, in non-relativistic limit $E_i \simeq m_i$
& at equilibrium $n_p^0 = e^{-m_p/T}$ and $n_n^0 = e^{-m_n/T}$

$$\frac{n_n^0}{n_p^0} = e^{-(m_n - m_p)/T} \qquad m_n - m_p = 1.29\ MeV$$

Therefore $n_n^0 \longrightarrow 0$

With expansion: annihilation processes freeze-out
Note: decay does not freeze-out (no free primordial neutrons today)

(2) for example: starting point

$$\frac{1}{a^3}\frac{d(n_n a^3)}{dt} = n_n^0 n_{\nu e}^0 \langle \sigma v \rangle \left(\frac{n_p n_{e^-}}{n_p^0 n_{e^-}^0} - \frac{n_n n_{\nu e}}{n_\nu^0 n_{\nu e}^0} \right)$$

Leptons e^-, ν_e are still in equilibrium since much lighter than n & p $\Rightarrow n_e = n_e^0$ & $n_\nu = n_\nu^0$

Righ-hand side $\Longrightarrow \Gamma_{np}\left(\frac{n_p n_n^0}{n_p^0} - n_n \right)$ where $\Gamma_{np} = n_{\nu e}^0 \langle \sigma v \rangle$

Change of variables : $X_n = \dfrac{n_n}{n_n + n_p}$

$$\frac{n_p}{n_n + n_p} = 1 - X_n$$

T function of time : radiation domination &

$$\rho = g_* \frac{\pi^2}{30} T^4 \quad , \quad \rho \propto a^{-4} \Longrightarrow T \propto a^{-1}$$

$$g_* = 10{,}75$$

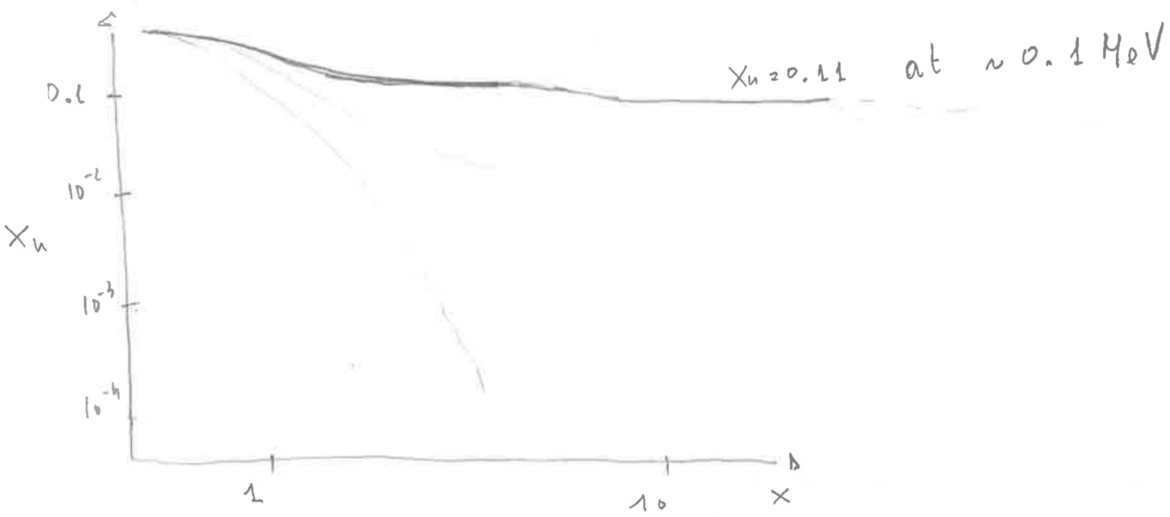
Change of variables : $x = \dfrac{m_n - m_p}{T} \propto a$

$$\frac{dx}{dt} = (m_n - m_p) \frac{\partial \left(\frac{1}{T}\right)}{\partial t} = - \frac{m_n - m_p}{T^2} \frac{\partial T}{\partial t} = - \frac{x}{T} \frac{\partial T}{\partial t}$$

$$= - \frac{x}{a^{-1}} \frac{\partial a^{-1}}{\partial t} = x \frac{\dot{a}}{a} = x H = x \sqrt{\frac{8 \pi G}{3} \rho}$$

$$\Longrightarrow \frac{d X_n}{d t} + 3 H X_n \sim \frac{\Gamma_{np}}{x H} \left[e^{-x} - X_n \left(1 + e^{-x} \right) \right]$$

$$\Gamma_{np} = \frac{255}{\tau_n x^5} \left(12 + 6x + x^2 \right)$$



Suppression factor : $X_n \to 0$ at late times.

Light elements formation

D & He through EM interactions

$$p + n \longrightarrow D + \gamma \qquad (1)$$

$$D + D \longrightarrow n + {}^3He \qquad (2)$$

$$^3He + D \longrightarrow p + {}^4He \qquad (3)$$

"Deuterium bottleneck" : nucleosynthesis through Deuterium formation
↳ D formed via (1) → apply Saha equation

$$\frac{n_D \, n_\gamma}{n_D^0 \, n_\gamma^0} = \frac{n_n \, n_p}{n_n^0 \, n_p^0}$$

Photons still at equilibrium at ~ 0.1 MeV : $n_\gamma = n_\gamma^0$

$$\frac{n_D}{n_n n_p} = \frac{n_D^0}{n_n^0 n_p^0}$$

$$E_i \simeq m_i \Rightarrow \frac{n_D}{n_n n_p} = \frac{3}{4}\left(\frac{2\pi\, m_D}{m_n m_p T}\right)^{3/2} e^{B_D/T}$$

$$B_D = m_n + m_p - m_D = 2.22 \text{ MeV}$$

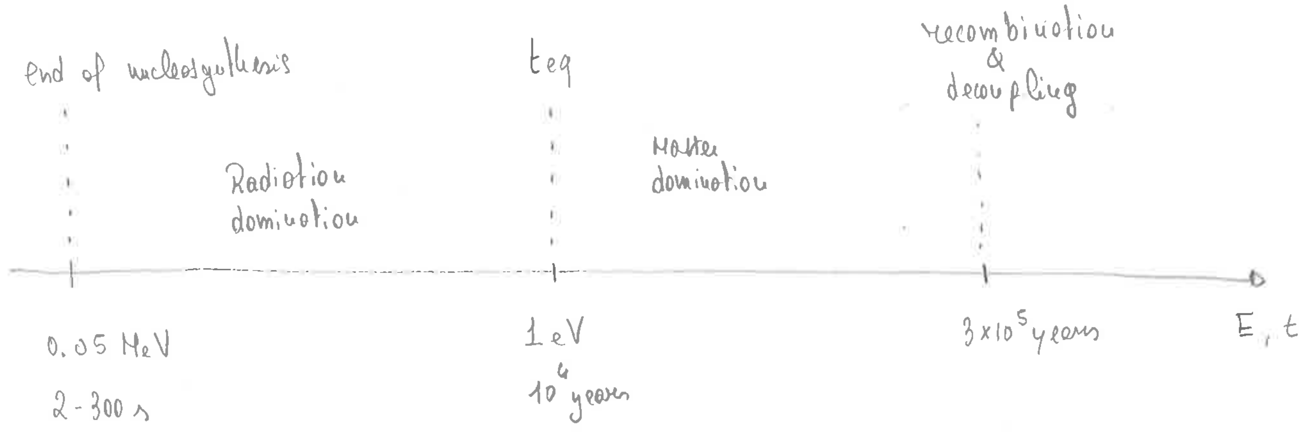
Rearrange as $\displaystyle \frac{n_D}{n_b} \sim \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{B_D/T}$

$$\eta_b = \frac{n_b}{n_\gamma} \simeq 10^{-9}$$

Assuming D forms instantaneously when $n_D \sim n_b$, find $T_{nuc} \simeq 0.07$ MeV
↳ all neutrons form ^4He $\Rightarrow X_4 = \dfrac{4 n_4}{n_b} = 2 X_n(T_{nuc})$

$$= 0.22$$

3.6 Recombination

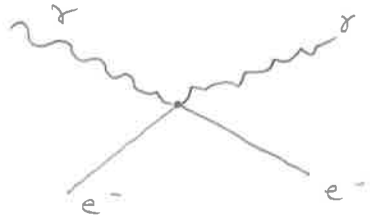


Recombination: free e^- combine with He^{2+} & H^+ to form neutral helium and hydrogen

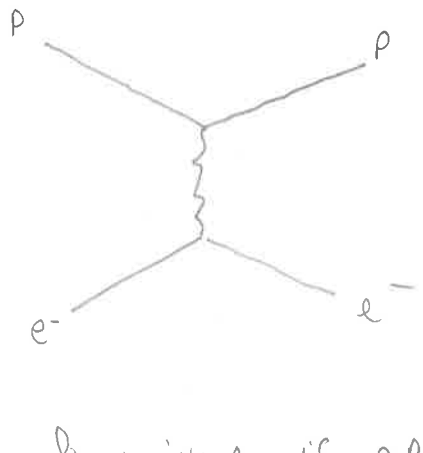
Helium recombination occurs before since He^{2+} has larger ionization potential.
Only after H^+ recombination universe becomes transparent to photons & CMB is released.

Consider H^+ recombination only for simplicity.

At t_{eq}, ~1 eV e^- still coupled to photons via Compton Scattering



p still coupled to e^- via Coulomb Scattering



Mean free path of γ is tiny & universe is opaque.

As long as $e^- + p \longrightarrow H + \gamma$

has rate $\Gamma \gg H$, evolution of number densities is given by Saha equation:

$$\frac{n_e \, n_p}{n_e^0 \, n_p^0} = \frac{n_\gamma \, n_H}{n_\gamma^0 \, n_H^0}$$

Assume $n_\gamma = n_\gamma^0 \implies \dfrac{n_e \, n_p}{n_H} = \dfrac{n_e^0 \, n_p^0}{n_H^0}$

$$E_i \simeq m_i \implies n_i^0 = g_i \exp\left(-m_i/T\right)\left(\frac{m_i T}{2\pi}\right)^{3/2}$$

$g_e = 2$, $g_p = 2$, $g_H = 4$

$$\frac{n_e \, n_p}{n_H} = e^{-\varepsilon_H/T}\left(\frac{m_e \cancel{m_p} T}{2\pi \cancel{m_H}}\right)^{3/2}$$

$$\simeq e^{-\varepsilon_H/T}\left(\frac{m_e T}{2\pi}\right)^{3/2}$$

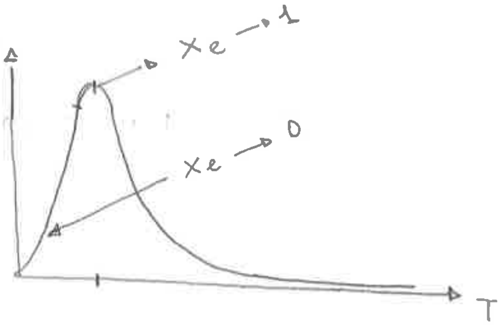
where $\varepsilon_H = m_p + m_e - m_H = 13.6 \text{ eV}$

On observable scales, the universe has zero net charge so assume $n_e = n_p$ and change variable to $X_e = \dfrac{n_e}{n_e + n_H} = \dfrac{n_p}{n_p + n_H}$

$$\implies \frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H}\left[\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\varepsilon_H/T}\right]$$

$$n_e + n_H = n_b = \eta_b \, n_\gamma \simeq 10^{-9} \, T^3$$

$$\implies \frac{X_e^2}{1 - X_e} \simeq 10^9 \left(\frac{m_e}{T 2\pi}\right)^{3/2} e^{-\varepsilon_H/T}$$



Equilibrium description fails before $X_e \to 0$.

In fact, the equilibrium description fails almost immediately after beginning of recombination.

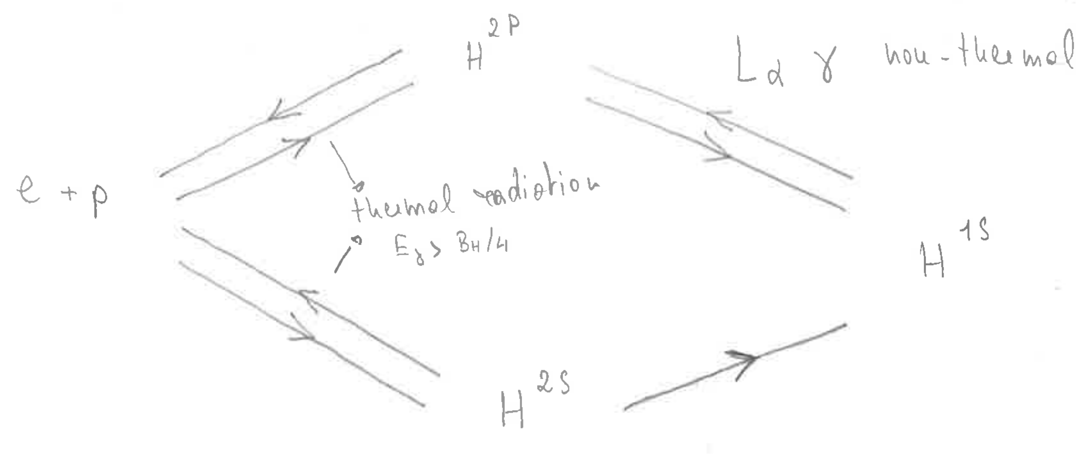
The main reason for this failure is the large number of γ emitted when p & e^- combine. (also $He^{2+} + 2e^-$).

This creates a non-thermal bath of photons that invalidates the equilibrium description.

One has to solve the BE instead.

The emitted photon in direct recombination $e^- + p \to H + \gamma$ has enough energy to immediately ionize another hydrogen atom. Therefore this process results in no net change in n_H.

More efficient is cascading recombination, where neutral hydrogen is first produced in an excited state and then decays to ground state.



Find $X_e^P \simeq 1.6 \times 10^{-5} \dfrac{\sqrt{\Omega_m}}{\Omega_b}$ $\qquad \Omega_m = \Omega_b + \Omega_{CDM} \simeq 0.3$

$\Omega_b \simeq 0.04$

$\simeq 2 \times 10^{-4}$

Recombination & decoupling

Decoupling of γ happens during recombination.

Freeze out $\quad \Gamma_\gamma \sim H$

$\downarrow$

Compton scattering : $\Gamma \simeq n_e \sigma_T = X_e n_b \sigma_T$

$\sigma_T = 0.665 \times 10^{-24}\, cm^2$ is Thomson cross-section

Recombination : X_e decreases $\Rightarrow \Gamma$ decreases $\Rightarrow$ decoupling happens earlier.

$$H^2 \simeq H_0^2 \left[\Omega_m a^{-3} + \Omega_\gamma a^{-4} \right] \quad \text{(neglect } \Lambda\text{)}$$

$$n_b \simeq 10^{-9}\, T^3 = 10^{-9}\, T_0^3\, a^{-3}$$

$$\Gamma \leq H \Rightarrow X_e \leq \frac{10^9 a^3}{T_0^3} H_0 \left[\Omega_m a^{-3} + \Omega_\gamma a^{-4} \right]^{+\frac{1}{2}}$$

Using $X_e(T)$ from recombination analysis, find $X_e \leq 10^{-2}$

4. Cosmological inflation

Observed universe: well described by flat ($K=0$) FRW model ΛCDM.

Early stages: radiation domination followed by matter domination

Issues: initial conditions $\to$ ΛCDM is fine-tuned.

4.1 The initial conditions problems

4.1.1 flatness problem

$\Omega_{tot} \simeq 1 + \Omega_K \simeq 1$.

Observational bounds $\quad \Omega_K = -0.037^{+0.043}_{-0.049}$

We know $\quad \Omega_{K_{today}} = \dfrac{\rho_K}{\rho_c} = -\dfrac{3K}{8\pi G}\dfrac{1}{\rho_c a^2}$

Unless $K=0$, Ω_K evolves

$$\begin{array}{ll} \text{radiation}: & \rho_c \propto a^{-4} \\ \text{matter}: & \rho_c \propto a^{-3} \end{array} \implies \Omega_K \propto \begin{cases} a^2 & \text{rad} \\ a & \text{matter} \end{cases}$$

Ω_K grows as universe expands.

ΛCDM: since Big Bang Ω_K grew of $\sim e^{60}$

$\implies$ must have been extremely small!

4.1.2 horizon problem

CMB highly homogeneous, $\frac{\delta T}{T} \sim 10^{-5}$ suggests thermalization.

ΛCDM: no mechanism for thermalization

Consider comoving particle horizon

$$\tau_{ph} = \int_{t_e}^{t} \frac{dt'}{a(t')}$$

$t_e = 0$ & rad $a \propto t^{1/2} \Rightarrow \tau_{ph} \propto t^{1/2}$

matter $a \propto t^{2/3} \Rightarrow \tau_{ph} \propto t^{1/3}$

Particle horizon grows $\Rightarrow$ observable universe becomes larger

$\Rightarrow$ smaller if $t \to -t$ $\Rightarrow$ photons of CMB never thermalized

$\frac{\delta T}{T} \sim 10^{-5}$ a coincidence?

Lecture 6.

4. Cosmological inflation

ΛCDM very successful in explaining observations of universe

However, it has issues with initial conditions.

4.1 Initial conditions problems

4.1.1 flatness problem

$$\Omega_{tot} = 1 + \Omega_K$$

$$\Omega_K = -0.037^{+0.043}_{-0.047}$$

Radiation & matter domination $\Rightarrow |\Omega_K|$ grows

Since hot Big Bang it grew by e^{60} $\Rightarrow$ fine tuning

4.1.1 horizon problem

$$\tau_{ph} = \int_0^t \frac{dt'}{a(t')} \propto \begin{cases} t^{1/2} & \text{radiation} \\ t^{1/3} & \text{matter} \end{cases} \Rightarrow \text{grows}$$

Causally connected patches were smaller in the past

$\Rightarrow$ no thermalization accross all observed sky

$\frac{\delta T}{T} \sim 10^{-5}$ a coincidence ?

4.2 Inflation: predicting initial conditions

To solve flatness problem: ρ must decrease slower than a^{-2}

To solve horizon problem: causally connected regions must have been larger in the past.

$$\Rightarrow \quad \tau_{ph} = \int_{\bar{t}_e}^{t} \frac{dt'}{a(t')} \quad \text{must be dominated by lower bound } \tau_{ph}(\bar{t}_e)$$

Causally connected region size is $(aH)^{-1}$. Require it decreases

$$\frac{d}{dt}(aH)^{-1} < 0 \quad \Rightarrow \quad \ddot{a} > 0$$

From acceleration equation this implies

$$\rho + 3p < 0 \quad \& \quad \rho > 0$$

$\Rightarrow$ form of energy with negative pressure.

Dark energy has this property, in fact $w < -\frac{1}{3}$ will do.

$\hookrightarrow$ need similar mechanism in primordial universe

Scalar field inflation

Energy density dominated by scalar field $\phi(t, \bar{x}) \rightarrow \phi(t)$, assume homogeneity for now.

Lagrangian density: $\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi)$

Noether theorem: $T_\nu{}^\mu = g^{\mu\alpha} \partial_\nu \phi \, \partial_\alpha \phi + g^\mu_\nu \mathcal{L}$

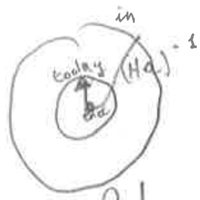
Identify $T_0^0 = -\rho_\phi$ & $T_0^i = P_\phi$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V$$

$$\rho + 3P < 0 \Rightarrow V(\phi) > \frac{1}{2}\dot{\phi}^2$$

4.2.1 How much inflation



Assume radiation domination throughout evolution.

$$a \propto t^{1/2} \quad \& \quad \dot{a} \propto t^{-1/2} \Rightarrow H \propto a^{-2}$$

and $$(aH)^{-1}\Big|_{end} \frac{1}{(aH)^{-1}}\Big|_{today} = \frac{a_0 H_0}{a_{end} H_{end}} = \frac{a_{end}}{a_0}$$

$T \propto \frac{1}{a}$ and $$\frac{1}{(aH)_{end}} = \frac{a_{end}}{a_0}\frac{1}{(aH)_0} = \frac{T_0}{T_{end}}\frac{1}{(aH)_0}$$

$T_0 \simeq 2.7\ K \simeq 2.35 \times 10^{-4}\ eV$

$T_{end} \geqslant 10^{10}\ GeV$

$$\Rightarrow \frac{1}{(aH)_{end}} \leq 10^{-23}\frac{1}{(aH)_0}$$

Observable universe at the end of inflation was at least 23 orders of magnitude smaller than today!

Minimal requirement to solve horizon problem: $(aH)^{-1}_{in} = (aH)^{-1}_0$

$$\Rightarrow \frac{a_{in}}{a_{end}} \geqslant 10^{23} \Rightarrow N = \ln\frac{a_{in}}{a_{end}} \geqslant 53$$

4.3 Slow-Roll

Inflation can be achieved in different ways.
Most popular realisation involves a scalar field slowly rolling towards vacuum.

Evolution of scalar field $\phi(t)$?

Use continuity equation

$$\dot{\rho} = -3H(\rho + p)$$

where
$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$
$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$\Rightarrow \quad \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0$$

If $\ddot{\phi} \ll 3H\dot{\phi}$ & $\dot{\phi}^2 \ll V(\phi)$, inflation happens.

Equation of motion reduces to

$$3H\dot{\phi} + \frac{\partial V}{\partial \phi} \simeq 0$$

This is slow-roll regime.

Conditions:

from def of inflation

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0 \quad \Rightarrow \quad \varepsilon \equiv -\frac{\dot{H}}{H^2} < 1$$

To neglect $\ddot{\phi}$ $\quad \Rightarrow \quad |\eta| \equiv \left| \frac{1}{H} \frac{\ddot{\phi}}{\dot{\phi}} \right| < 1$

ε & η are called the slow-roll parameters

Using SR EoM, they can be translated in terms of potential derivatives:

$$\varepsilon = \frac{1}{2} \frac{1}{8\pi G} \left(\frac{V_{,\phi}}{V} \right)^2$$

$$\eta = \frac{1}{8\pi G} \frac{V_{,\phi\phi}}{V}$$

4.3.1 Inflationary models

Model building $\longrightarrow$ choice of potential $V(\phi)$

The initial conditions ϕ_i & $\dot{\phi}_i$ have to be chosen such that the field is slow-rolling towards global minimum with H nearly constant.

Inflation ends when field oscillates around global minimum. Then ϕ decays into other particles, "reheating" the universe & radiation dominated expansion takes over.

Simple example: $V(\phi) = \frac{1}{2} m^2 \phi^2$

$$\Longrightarrow \quad \varepsilon = \frac{1}{4\pi G} \frac{1}{\phi^2}$$

$$\eta = \frac{1}{4\pi G} \frac{1}{\phi^2} = \varepsilon$$

Inflation ends when slow-roll conditions fail:

$$\varepsilon = \eta = 1 \implies \phi_{end} = \frac{1}{\sqrt{4\pi G}}$$

EoM $\implies$
$$H^2 \simeq \frac{4\pi G}{3} m^2 \phi^2$$
$$3H\dot{\phi} \simeq - m^2 \phi$$

More convenient to solve in terms of e-folds $N \equiv \ln a$

$$\frac{\partial}{\partial N} = \frac{\partial t}{\partial N}\frac{\partial}{\partial t} = \frac{a}{\dot{a}}\frac{\partial}{\partial t} = \frac{1}{H}\frac{\partial}{\partial t}$$

$$\implies \quad 3H^2 \frac{\partial \phi}{\partial N} = - m^2 \phi$$

$$\frac{\partial \phi}{\partial N} = - \frac{1}{4\pi G}\frac{1}{\phi} \implies \int_{\phi_i}^{\phi_{end}} \phi \, d\phi = -\int_{N_i}^{N_{end}} \frac{dN}{4\pi G}$$

$$\frac{\phi_{end}^2 - \phi_i^2}{2} = \frac{N_i - N_{end}}{4\pi G} = - \frac{\Delta N}{4\pi G}$$

Minimal inflation: $\Delta N \geqslant 64$ & $\phi_{end}^2 = \frac{1}{4\pi G}$

$$\implies \phi_i^2 \geqslant \frac{1}{4\pi G}\left(128 - 1\right)$$

Scenario usually referred to as chaotic inflation.

4.4 Quantum fluctuations during inflation

Inflation was motivated by flatness & horizon problems

However most important feature ⟹ mechanism for origin of structures of universe through quantum fluctuations.

Indeed ϕ is a quantum field.

↳ quantization in curved spacetime ⟹ problematic.

Canonical quantization : relies on global invariance of the metric under time transformations in order to have decomposition of Fourier modes into positive & negative frequency.

Path integral quantization : same problem.

Way around : on scales smaller than Hubble radius $(aH)^{-1}$, the FRW metric reduces to Minkowski.
(intuitively : short distances & time : $a \simeq cst$ & $\tau \ll \frac{1}{k}$)

⟹ on such scales vacuum fluctuations can be defined without ambiguity.

Then via inflation, the fluctuations are stretched to larger scales ⟹ eventually they exit the Hubble radius

After they are "frozen" : they are conserved until they reenter the horizon.

⟹ need to find out how fluctuations evolve until horizon exit.

To this end, we consider initial quantum fluctuations for scales well inside horizon (no gravity) and evolve them classically until horizon exit.

Scalar field $\phi(t,\bar{x}) = \phi(t) + \delta\phi(t,\bar{x})$

$$\Rightarrow \quad \delta\ddot{\phi}(t,\bar{x}) + 3H\delta\dot{\phi}(t,\bar{x}) + (\underbrace{-\nabla^2}_{\frac{1}{a^2}\sum_i \frac{\partial^2}{\partial x_i^2}} + m^2)\,\delta\phi(t,\bar{x}) = 0$$

In terms of Fourier modes: $\delta\phi(t,\bar{x}) = \int \frac{d^3k}{(2\pi)^3}\,\delta\phi(t,\bar{k})\,e^{i\bar{k}.\bar{x}}$

$$\Rightarrow \quad \delta\ddot{\phi}(t,\bar{k}) + 3H\delta\dot{\phi}(t,\bar{k}) + \left(\frac{k^2}{a^2} + m^2\right)\delta\phi(t,\bar{k}) = 0$$

During inflation $\ddot{a} > 0 \Rightarrow$ if it lasts enough, there exists time in the past when $\frac{k^2}{a^2} \gg m^2$ for all k relevant to cosmological observations. $\Rightarrow$ treat $\delta\phi$ as massless.

At tree level, quantum fluctuation are

$$\langle \delta\phi^*(t_i,\bar{k})\,\delta\phi(t_i,\bar{q})\rangle = \frac{1}{2k}\,(2\pi)^3\,\delta^{(3)}(\bar{k}+\bar{q})$$

$$\langle \delta\dot{\phi}^*(t_i,\bar{k})\,\delta\dot{\phi}(t_i,\bar{q})\rangle = \frac{k}{2}\,(2\pi)^3\,\delta^{(3)}(\bar{k}+\bar{q})$$

Assuming $\frac{k^2}{a^2} \gg m^2$ until horizon crossing $(k = aH)$

and $H = cst \Rightarrow \quad \delta\ddot{\phi}(t,\bar{k}) + 3H\delta\dot{\phi}(t,\bar{k}) + \frac{k^2}{a^2}\delta\phi(t,\bar{k}) = 0$

In terms of conformal time $\tau = \int \frac{dt}{a} = -\frac{1}{aH}$ if $H = cst$

$$\Rightarrow \frac{d^2\delta\phi(\tau,\bar{k})}{d\tau^2} + \left(k^2 - \frac{2}{\tau^2}\right)\delta\phi(\tau,\bar{k}) = 0$$

The solution such that at $t \to -\infty$, $\tau \to -\infty$ we recover Minkowski vacuum is

$$\delta\phi(\tau, \bar{k}) = \frac{e^{-ik\tau}}{\sqrt{2k}} \frac{(k\tau - i)}{k\tau}$$

$$\langle \delta\phi^*(\tau, \bar{k})\, \delta\phi(\tau, \bar{q}) \rangle = (2\pi)^3 \delta^{(3)}(\bar{k} + \bar{q})\, |\delta\phi(\tau, \bar{k})|^2$$

$$= (2\pi)^3 \delta^{(3)}(\bar{k} + \bar{q}) \frac{1}{2k^3\tau^2} (k^2\tau^2 + 1)$$

$$= (2\pi)^3 \delta^{(3)}(\bar{k} + \bar{q}) \left(\frac{1}{2k} + \frac{a^2 H^2}{2k^3} \right)$$

For $k \gg aH$ we recover Minkowski result as expected.

However for $k \ll aH$ result is strikingly different

Power spectrum: $P_\phi(k, \tau) \equiv \frac{k^3}{2\pi^2} |\delta\phi(\tau, k)|^2$

At horizon crossing find

$$P_\phi(k, \tau) = \left(\frac{H}{2\pi} \right)^2$$

Lecture 7

4.4 quantum fluctuations during inflation

$$\phi(t,\bar{x}) = \phi(t) + \delta\phi(t,\bar{x})$$

$$\Rightarrow \text{EOM}: \quad \delta\ddot{\phi}(t,\bar{x}) + 3H\,\delta\dot{\phi}(t,\bar{x}) + (-\underbrace{\nabla^2}_{\frac{1}{a^2}\sum_i \frac{\partial}{\partial x_i}} + m^2)\,\delta\phi(t,\bar{x}) = 0$$

$$m^2 = \left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=0}$$

go to Fourier space $\quad \delta\phi(\bar{x},t) = \int \frac{d^3k}{(2\pi)^3}\, \delta\tilde{\phi}(t,\bar{k})\, e^{i\bar{k}\cdot\bar{x}}$

$$\Rightarrow \delta\ddot{\phi}(t,\bar{k}) + 3H\,\delta\dot{\phi}(t,\bar{k}) + \left(\frac{k^2}{a^2} + m^2\right)\delta\phi(t,\bar{k}) = 0$$

Inflation: $\ddot{a} > 0 \Rightarrow$ if lasts long enough $\frac{k^2}{a^2} \gg m^2$ in the past.

Thus treat field as massless.

Minkowski vacuum:

$$\langle 0|\,\delta\hat{\phi}^*(t_i,\bar{q})\,\delta\hat{\phi}(t_i,\bar{k})\,|0\rangle = \frac{1}{2k}(2\pi)^3\,\delta^3(\bar{k}+\bar{q})$$

Indeed: $\quad \delta\phi(t_i,\bar{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\omega_k t_i}}{2\omega_k}\left(a_k e^{-i\bar{k}\bar{x}} + a_k^\dagger e^{i\bar{k}\bar{x}}\right)$

$$\Rightarrow \delta\hat{\phi}(t_i,\bar{k})|0\rangle = \frac{e^{i\omega_k t_i}}{2\omega_k}a_k^\dagger|0\rangle \qquad \omega_k^2 = |k|^2 + m^2 = |k|^2$$

$$= \frac{e^{i\omega_k t_i}}{2\omega_k}|k\rangle$$

$$\langle 0|\,\delta\hat{\phi}^*(t_i,\bar{q}) = \frac{e^{-i\omega_q t_i}}{2\omega_k}\langle 0|a_q = \frac{e^{-i\omega_k t_i}}{2\omega_k}\langle q|$$

$$\Rightarrow \langle 0|\,\delta\hat{\phi}^*(t_i,\bar{q})\,\delta\hat{\phi}(t_i,\bar{k})\,|0\rangle = \frac{e^{i(\omega_k-\omega_q)t_i}}{4\omega_k\omega_q}\langle q|k\rangle \quad \& \quad \langle q|k\rangle = (2\pi)^3\,2\omega_q\,\delta^3(\bar{q}+\bar{k})$$

$$= e^{i(\omega_k-\omega_q)t_i}\frac{(2\pi)^3}{2\omega_k}\delta^3(\bar{k}+\bar{q})$$

because of $\delta^3 \to e^{i(\omega_k - \omega_q) t_i} = 1$

Evolve $\delta \hat{\phi}(t_i, \bar{k})$ classically & $t_i \to -\infty$

$$\delta \hat{\phi}(t, \bar{k}) = \underbrace{\delta \phi(t, \bar{k})}_{\text{solution of FRW equations of motion}} \delta \hat{\phi}(-\infty, \bar{k})$$

Assume $\frac{k^2}{a^2} \gg m^2$ until horizon crossing at $k = aH$ & $H = \text{cst}$

Use conformal time: $\tau = \int \frac{dt}{a} = -\frac{1}{aH}$

$$\Rightarrow \frac{d^2 \delta \phi(\tau, \bar{k})}{d\tau^2} + \left(k^2 - \frac{2}{\tau^2} \right) \delta \phi(\tau, \bar{k}) = 0$$

Solution is $\delta \phi(\tau, \bar{k}) = e^{-ik\tau} \frac{k\tau - i}{k\tau}$ | different treatment see Mukhanov 340 - 343

Find $\langle 0 | \delta \hat{\phi}^*(\tau, \bar{q}) \delta \hat{\phi}(\tau, \bar{k}) | 0 \rangle = \delta \phi(\tau, \bar{k}) \delta \phi^*(\tau, \bar{q}) \langle 0 | \delta \hat{\phi}^*(-\infty, \bar{q}) \delta \hat{\phi}(-\infty, \bar{k}) | 0 \rangle$

$$= (2\pi)^3 \delta^3(\bar{k} + \bar{q}) \left(\frac{1}{2k} + \frac{a^2 H^2}{2k^3} \right)$$

$$\mathcal{P}_\phi(k, \tau) \equiv \frac{k^3}{2\pi^2} | \delta \phi(\tau, k) |^2$$

and at horizon crossing

$$\mathcal{P}_\phi(k, \tau) \Big|_{k = aH} = \left(\frac{H_k}{2\pi} \right)^2$$

4.5 Curvature Perturbation

FRW metric is homgeneous + isotropic, good approximation on large scales.

However on scales we experience universe is inhomogeneous.

More realistic description: FRW + perturbations.

In what follows we introduce first order perturbations of FRW & of $T_\mu{}^\nu$

4.5.1 1st order metric perturbations

$$ds^2 = \left[g_{\mu\nu} + \delta g_{\mu\nu} \right] dx^\mu dx^\nu \quad \text{with} \quad |g_{\mu\nu}| \gg |\delta g_{\mu\nu}|$$

In terms of conformal time τ

$$g_{\mu\nu} dx^\mu dx^\nu = a^2(\tau) \left(-d\tau^2 + d\Sigma^2 \right)$$

Perturb to 1st order:

$$\delta g_{00} = -2 a^2(\tau) \, A(\tau, \bar{x})$$

$$\delta g_{0i} = a^2(\tau) \left[\partial_i B(\tau, \bar{x}) + S_i(\tau, \bar{x}) \right]$$

$$\delta g_{ij} = a^2(\tau) \Big[-2\psi(\tau, \bar{x}) \delta_{ij} + 2 \partial_i \partial_j E(\tau, \bar{x})$$
$$+ \partial_i F_j(\tau, \bar{x}) + \partial_j F_i(\tau, \bar{x}) + h_{ij} \Big]$$

$$i, j = 1, 2, 3$$

We have

- Scalar perturbations A, B, ψ, E. They are at origin of structures in universe

- vector perturbations S_i, F_i. They decay very quickly and usually to not lead to cosmological signatures.

- Tensor perturbations h_{ij} correspond to gravitational waves. Although generated during inflation, they have low amplitude & no important effects on structure formation.

Scalar, vector & tensor perturbations are decoupled & can be studied separately.

4.5.2 1st order $T_\mu^{\ \nu}$ perturbations

$$T_\mu^{\ \nu} = (\rho + P)\, u^\nu u_\mu + \rho\, \delta_\mu^\nu$$

$$T_\mu^{\ \nu} \to T_\mu^{\ \nu} + \delta T_\mu^{\ \nu}$$

$$\delta T_0^{\ 0} = -\delta\rho$$

$$\delta T_i^{\ 0} = (\rho + P)(\partial_i \delta u + \delta u_i^{v})$$

$$\delta T_0^{\ i} = \frac{\rho + P}{a(\tau)^2}\left(a(\tau)\, \partial_i B(\tau, \bar{x}) + a(\tau)\, S_i(\tau, \bar{x}) - \partial_i \delta u - \delta u_i^{v}\right)$$

$$\delta T^i_j = \delta^i_j \delta p + \partial^i \partial_j \pi^S + \partial^i \pi^V_j + \pi^i_j{}^T$$

$$\delta T^\lambda_\lambda = 3\delta p - \delta \rho + \nabla^2 \pi^S$$

From Einstein equations :

$$G^\mu_\nu + \delta G^\mu_\nu = T^\mu_\nu + \delta T^\mu_\nu$$

$$\Rightarrow \quad \delta G^\mu_\nu = \delta T^\mu_\nu \quad \Rightarrow \text{ evolution eqs. for perturbations}$$

9.5.3 Gauge transformations

How are perturbations affected by coordinate transformations

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$$

for ξ^μ infinitesimal ?

At given spacetime point, metric tensor transforms at linear order as

$$\tilde{g}_{\mu\nu}(\tilde{x}^\gamma) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \left(g_{\alpha\beta}(x^\gamma) + \delta g_{\alpha\beta}(x^\gamma) \right)$$

$$\simeq g_{\mu\nu}(x^\gamma) + \delta g_{\mu\nu}(x^\gamma) - g_{\mu\beta}(x^\gamma) \frac{\partial \xi^\beta}{\partial \tilde{x}^\nu} \quad g_{\alpha\nu}(x^\gamma) \frac{\partial \xi^\alpha}{\partial \tilde{x}^\mu}$$

Metric scalar perturbations become :

$$A \to \tilde{A} = A - \frac{1}{a}(a\xi^0)' \qquad \text{where} \quad \xi^\mu = (\xi_0, \xi^i)$$

$$B \to \tilde{B} = B + k' - \xi^0 \qquad \& \quad \xi^i = \xi^i_\perp + \frac{\partial k}{\partial x_i}$$

$$\psi \to \tilde{\psi} = \psi + \frac{a'}{a}\xi_0$$

$$E \to \tilde{E} = E + k \qquad \text{and} \quad ' = \frac{\partial}{\partial \tau}$$

Combinations that do not depend on coordinate transformations are called gauge invariant. They are good observables

The simplest are

$$\Phi \equiv A - \frac{1}{a}\left[a\,(B-E')\right]'$$

$$\Psi \equiv -\psi + \frac{a'}{a}\,(B-E')$$

We have freedom to choose $\xi^\mu \Rightarrow$ choose k & ξ^0

4.5.4 Newtonian gauge & curvature perturbation

Choose ξ^0 & k so that $B = E = 0$

This is Newtonian gauge & scalar perturbations are reduced to

$$\delta g_{00} = -2a(\tau)^2 A(\tau,\bar{x})$$

$$\delta g_{ij} = -2a(\tau)^2 \psi(\tau,\bar{x})\,\delta_{ij}$$

$\psi = -\Psi$ for this gauge, is of particular interest since it describes perturbations on constant time hypersurfaces $\Rightarrow$ direct relation to CMB measurements.

The evolution of ψ is given by perturbed Einstein eqs:

$$3(\rho+p)\dot{\psi} = \delta\dot{\rho} + 3\frac{\dot{a}}{a}(\delta\rho+\delta p) + \nabla^2\left[\frac{\rho+p}{a^2}\delta u + \frac{\dot{a}}{a}\pi^S\right]$$

Consider ψ evaluated on constant energy density

$$\zeta = -\psi|_{\delta\rho=0}$$

$\hookrightarrow$ curvature perturbation.

On superhorizon scales $k < aH$, ∇^2 is negligible &

$$\dot{\zeta} = -\dot{\psi}|_{\delta\rho=0} = -\frac{H}{\rho+p}\delta p \quad \text{superhorizon.}$$

If the universe is dominated by a single fluid then $\delta p = 0$ and ζ is conserved!

Relation of ζ to slow-roll inflation

$$da^2 = 2a^2\psi \quad \& \quad da^2 = 2a\,da \Rightarrow \frac{da}{a} = \frac{da^2}{2a^2} = \psi$$

$$\text{SR}: \quad 3H\dot{\phi} \simeq -V_{,\phi} \Rightarrow 3H\frac{\partial N}{\partial t}\frac{\partial\phi}{\partial N} \simeq -V_{,\phi}$$

$$\Rightarrow \frac{\partial\phi}{\partial N} \simeq -\frac{V_{,\phi}}{3H}\frac{1}{H} \simeq -\frac{\dot{\phi}}{H}$$

$$dN = d(\ln a) = \frac{da}{a} = -\frac{H}{\dot{\phi}}d\phi$$

$$\Rightarrow \zeta = \left(\frac{H}{\dot{\phi}}\delta\phi\right)_{k=aH}$$

The Power spectrum is then simply given by replacing $\delta\phi$ with P_ϕ

$$P_\zeta = \left[\left(\frac{H}{\dot\phi} \right)^2 P_\phi \right]_{k=aH}$$

$$= \left[\left(\frac{H}{\dot\phi} \right)^2 \left(\frac{H}{2\pi} \right)^2 \right]_{k=aH}$$

Assume $P_\zeta \propto k^{n_s - 1}$ where n_s is spectral index

If $n_s = 1 \Rightarrow$ scale invariant

Planck data : $n_s = 0.9603 \pm 0.0073$ at 68 % C.L.

No fundamental reason to expect P_ζ to follow power law.

However, we can define effective spectral index

$$n_s(k) - 1 = \frac{d \ln P_\zeta}{d \ln k}$$

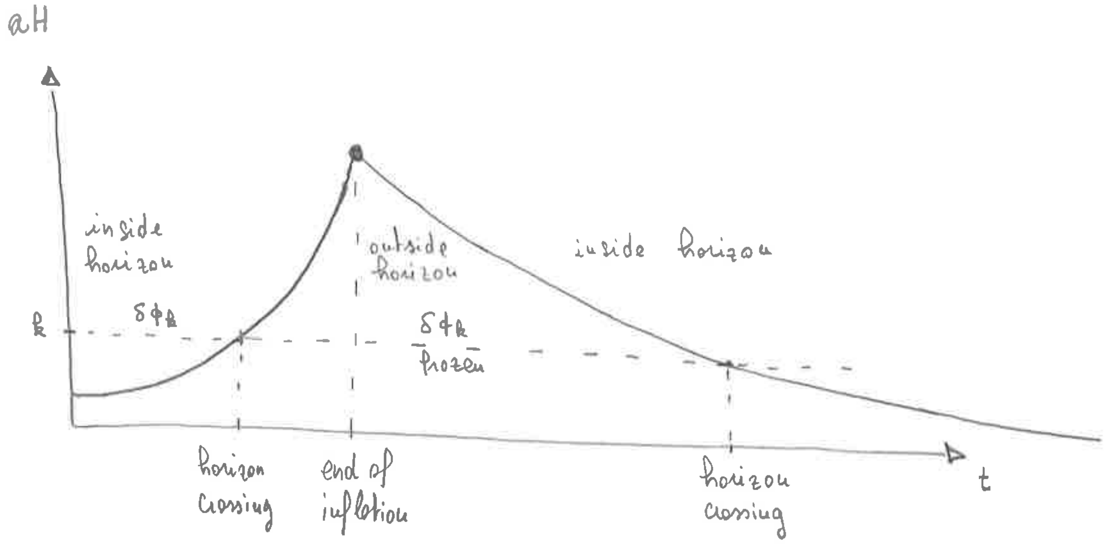
$$P_\zeta = \frac{9H^6}{V_{,\phi}^2} \frac{1}{4\pi^2} \simeq \frac{1}{12\pi^2 m_{pl}^2} \frac{V^3}{V_{,\phi}^2} = \frac{1}{24\pi^2 m_{pl}^2} \frac{V}{\varepsilon}$$

$$(n_s - 1)\Big|_{k=aH} = -6\varepsilon_* + 2\eta_*$$

where $*$ indicates evaluation at horizon crossing.

Lecture 8

Last lecture: related quantum fluctuations of inflaton field to metric perturbations $\rightarrow$ curvature perturbations.



Relation between P_ϕ & P_ζ :- perturb Einstein equations

$$G_{\mu\nu} + \delta G_{\mu\nu} = T_{\mu\nu} + \delta T_{\mu\nu}$$

Unperturbed quantities balance each other &

$$\delta G_{\mu\nu} = \delta T_{\mu\nu}$$

- Go to Newtonian Gauge s.t.

$$\delta g_{ij} = -2a^2(\tau)\,\psi(\tau,\bar{x})\,\delta_{ij} \quad \text{(neglecting vector \& tensor)}$$

Define $\zeta = -\psi\big|_{\delta\rho=0}$ as curvature perturbation.

- For $k < aH$ $\quad \dot{\zeta} = -\dfrac{H}{\rho+p}\,\delta p$ is conserved when $\delta p = 0$

- write $\delta g_{ij} = -2a^2(\tau)\,\psi(\tau,\bar{x})$ as

$$da^2 = 2a^2\psi \quad \text{using} \quad \frac{\partial g_{ij}}{\partial a^2}\,da^2 = \delta g_{ij}$$

$$\Rightarrow \frac{da^2}{2a^2} = \frac{da}{a} = \psi$$

during slow-roll $3H\dot{\phi} \simeq -V_{,\phi} \Longrightarrow \frac{\partial \phi}{\partial N} \simeq -\frac{V_{,\phi}}{3H^2} \simeq -\frac{\dot{\phi}}{H}$

& $dN = d\ln a = \frac{da}{a} = -\frac{H}{\dot{\phi}} d\phi$

$$\Longrightarrow \zeta = \left(\frac{H}{\dot{\phi}}\delta\phi\right)_{k=aH}$$

Power spectrum : $\mathcal{P}_\zeta = \left[\left(\frac{H}{\dot{\phi}}\right)^2 \mathcal{P}_\phi\right]_{k=aH}$

$$= \left[\left(\frac{H}{\dot{\phi}}\right)^2\left(\frac{H}{2\pi}\right)^2\right]_{k=aH}$$

Scale dependence of $\mathcal{P}_\zeta$: assume $\mathcal{P}_\zeta \propto k^{n_s - 1}$

Planck data : $n_s = 0.9603 \pm 0.0073$ at 68 % CL. at $k_* = 0.002\, h\,\mathrm{Mpc}^{-1}$

$h = 100\, h$ km/s/Mpc

No reason to expect $\mathcal{P}_\zeta$ to follow power law

Define effective spectral index

$$n_s(k) - 1 = \frac{d\ln \mathcal{P}_\zeta}{d\ln k}$$

$$\mathcal{P}_\zeta = \frac{9H^6}{V_{,\phi}^2}\frac{1}{4\pi^2} \simeq \frac{1}{12\pi^2 m_{pl}^2}\frac{V^3}{V_{,\phi}^2} = \frac{1}{24\pi^2 m_{pl}^2}\frac{V}{\varepsilon}$$

$$(n_s - 1)\big|_{k=aH} = -6\varepsilon_* + 2\eta_*$$

Derivation : $\left.\frac{d \ln P_S}{d \ln k}\right|_{k=aH} = \frac{\varepsilon}{V} \frac{d \frac{V}{\varepsilon}}{d \ln k} = \frac{aH\varepsilon}{V} \frac{d \frac{V}{\varepsilon}}{d(aH)}$ ②

$$= \frac{aH\varepsilon}{V} \frac{d\phi}{d(aH)} \frac{d \frac{V}{\varepsilon}}{d\phi}$$

$$= \frac{aH\varepsilon}{V} \frac{dt}{d(aH)} \frac{d\phi}{dt} \frac{d \frac{V}{\varepsilon}}{d\phi}$$

$$\frac{d \frac{V}{\varepsilon}}{d\phi} = \frac{V_{,\phi}}{\varepsilon} - \frac{V}{\varepsilon^2} \frac{d\varepsilon}{d\phi}$$

$$\frac{d\varepsilon}{d\phi} = \frac{m_{pe}^2}{2} \frac{d}{d\phi} \left(\frac{V_{,\phi}}{V} \right)^2 = m_{pe}^2 \frac{V_{,\phi}}{V} \left(\frac{V_{,\phi\phi}}{V} - \frac{V_{,\phi}^2}{V^2} \right)$$

$$= \frac{V_{,\phi}}{V} \left(\eta - 2\varepsilon \right)$$

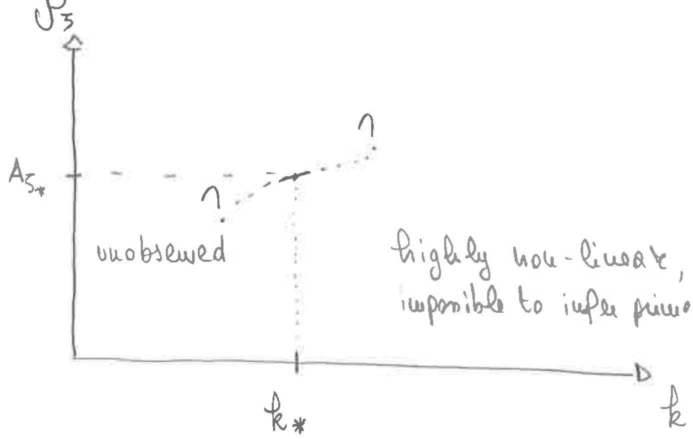
$$\frac{d \frac{V}{\varepsilon}}{d\phi} = \frac{V_{,\phi}}{\varepsilon} - \frac{V_{,\phi}}{\varepsilon^2} \left(\eta - 2\varepsilon \right) = \frac{V_{,\phi}}{\varepsilon^2} \left(3\varepsilon - \eta \right)$$

$$\frac{d\phi}{dt} \simeq - \frac{V_{,\phi}}{3Hm_{pe}^2} \quad \& \quad \frac{dt}{daH} = \frac{1}{\ddot{a}} \quad \& \quad \ddot{a} = -\frac{4\pi G a}{3}(\rho + 3p) \quad \rho \simeq V, \; p \simeq -V$$
$$\simeq \frac{Va}{3m_{pe}^2}$$

$$\left.\frac{d \ln P_S}{d \ln k}\right|_{aH} = -\frac{\cancel{aH}\cancel{\varepsilon}}{V} \frac{\cancel{3}m_{pe}^{\cancel{2}}}{V\cancel{a}} \frac{V_{,\phi}}{\cancel{3H}\cancel{m_{pe}^2}} \frac{V_{,\phi}}{\varepsilon^{\cancel{2}}} \left(3\varepsilon - \eta \right) \Big|_{aH}$$

$$= \frac{V_{,\phi}^2}{V^2} \frac{1}{\varepsilon} \left(\eta - 3\varepsilon \right) \Big|_{aH}$$

$$\boxed{n_s - 1 = 2\eta_* - 6\varepsilon_*}$$



Around k_* :

$$\mathcal{P}_S \simeq A_{S*}\left(\frac{k}{k_*}\right)^{n_S-1}$$

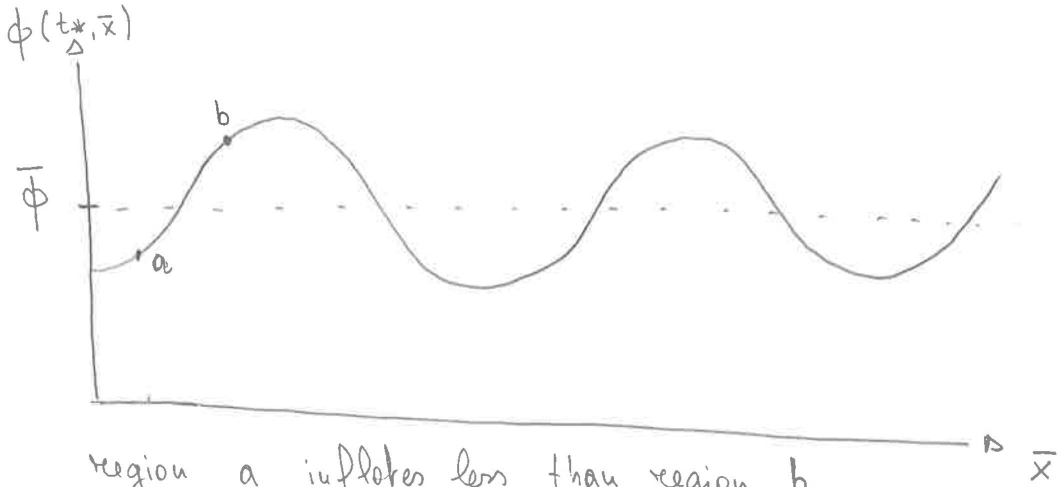
A_{S*} & n_S are measured

$A_{S*} = 2.3\times10^{-9}$

In terms of inflationary parameters $A_{S*} = \dfrac{1}{24\pi^2 m_{Pl}^2} \dfrac{V}{\varepsilon}\Big|_{k=aH}$

Inflation relates $\mathcal{P}_S$ to values of inflationary parameters at horizon crossing $k = aH$.

For single field model



region a inflates less than region b

$\Rightarrow$ $\Delta a = a_a - a_b$ sources curvature perturbation

Example : $V = \frac{m^2 \phi^2}{2}$

Already calculated $\varepsilon = \frac{2 m_{pl}^2}{\phi^2}$, $\eta = \frac{2 m_{pl}^2}{\phi^2}$

$\phi_{end} = \sqrt{2}\, m_{pl}$

$$\frac{\phi_*^2 - \phi_{end}^2}{2} = 2 m_{pl}^2 \Delta N_* = 128 m_{pl}^2$$

$$\phi_* = \sqrt{256 - 2}\; m_{pl} \simeq 16\, m_{pl}$$

$$\varepsilon_* = \frac{2 m_{pl}^2}{254 m_{pl}^2} = \frac{1}{127} = \eta_*$$

$$n_s - 1 = -6 \varepsilon_* + 2\eta_* = -\frac{4}{127} \simeq -0.0315$$

$n_s = 0.9685 \Rightarrow$ within 68% C.L.

$$A_{s*} = \frac{1}{24\pi^2 m_{pl}^2} \frac{m^2 \phi_*^2}{4 m_{pl}^2} \phi_*^2 = \frac{m^2 \phi_*^4}{96 \pi^2 m_{pl}^4} \simeq 2.3 \times 10^{-9}$$

$$\Rightarrow m^2 \simeq \frac{2.3 \times 10^3 \times 10^{-9}}{0.95 \times 6.5 \times 10^4} m_{pl}^2 = \frac{10^{-10}}{6.17} m_{pl}^2 \simeq 3.8 \times 10^{-10}\, m_{pl}^2$$

All parameters determined by observations

However, model is ruled out by tensor to scalar ratio

$$\frac{P_h}{P_s} = r = 16\, \varepsilon_*$$

P_h is power spectrum of gravity waves sourced during inflation by quantum fluctuations of h_{ij}.

4.6 Multi-field inflation

No reason to expect inflation to be sourced by a single scalar field.
In fact, inflationary models issued from particle physics generally contain many scalar fields.

$$V(\phi_i) \quad , \qquad i = 1, \dots, q$$

During slow-roll, all fields obey the slow-roll EoM:

$$3H \frac{\partial \phi_i}{\partial t} \simeq - \frac{\partial V}{\partial \phi_i} \frac{1}{m_{pl}^2}$$

We then have many slow-roll parameters:

$$\varepsilon_i = \frac{m_{pl}^2}{2} \left(\frac{V_{,i}}{V} \right)^2 \qquad V_{,i} = \frac{\partial V}{\partial \phi_i}$$

$$\eta_{ij} = m_{pl}^2 \frac{V_{,ij}}{V} \qquad V_{,ij} = \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}$$

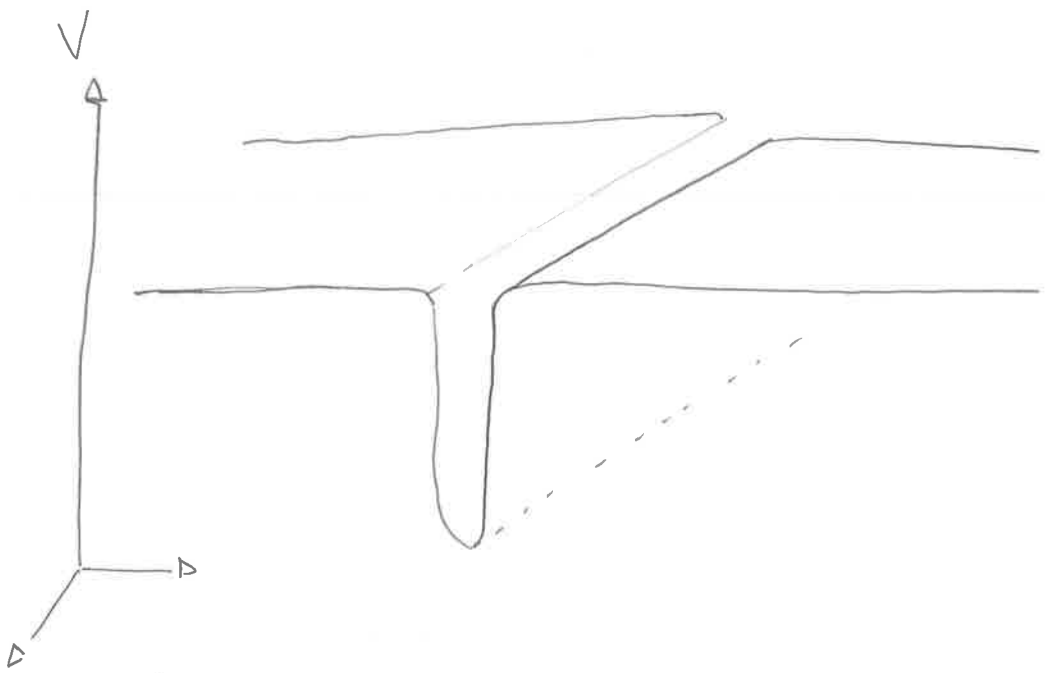
We have $\mathcal{P}_5 = \frac{1}{24 m_{pl}^2} \sum_i \frac{V}{\varepsilon_i}$

Could go through same derivation as before & compute $n_s|_{k=aH}$

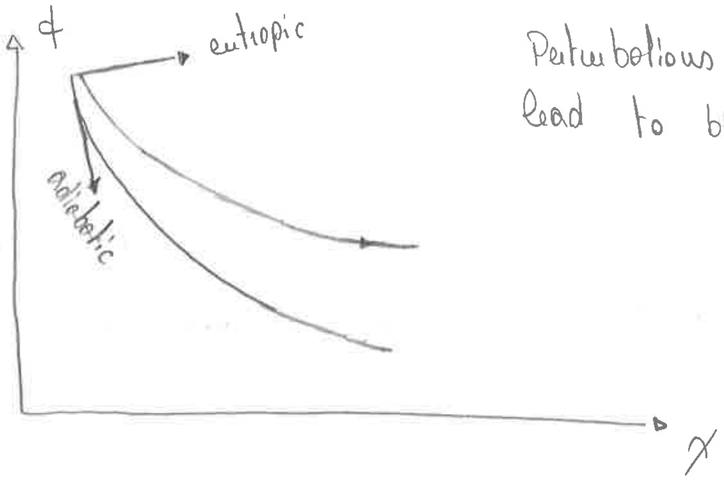
<u>However</u> $\dot{\zeta} \neq 0$ at horizon crossing now since we have more than one field and we can't assume $\delta P = 0$.

Therefore to tackle general multi-field models we need a new formalism, able to track the evolution of ζ until it is conserved.

Exceptions are multi-field models that are effectively single field: one light field & the other fields are heavy with $|m_i^2| > H^2$.



In general (consider 2 fields)



Perturbations in entropic direction can lead to big difference in trajectories.

If trajectory in field space is a straight line the multi-field model is really a single field one (can rotate in field space & describe everything in terms of one field)

4.7 Initial conditions for inflation

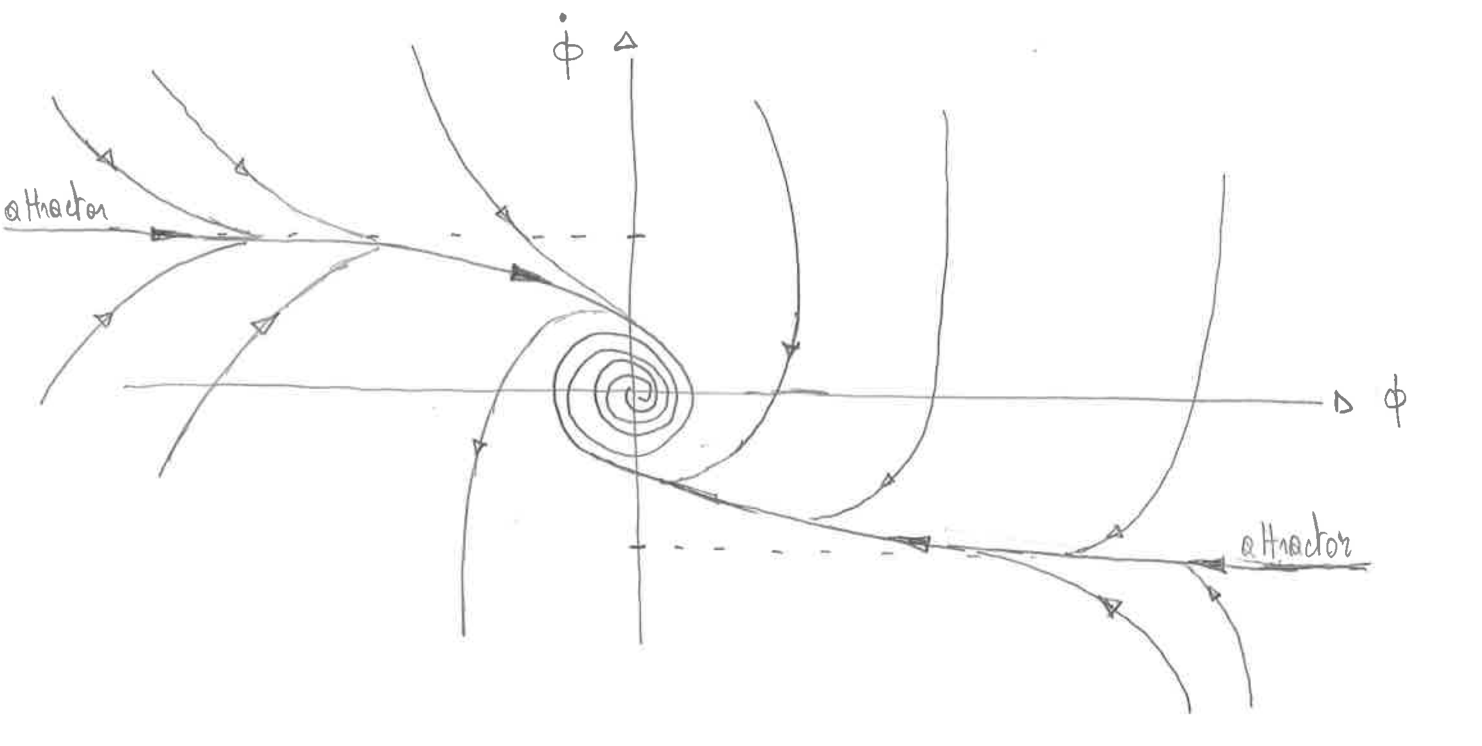
Inflation was introduced to address the issue of initial conditions of the Hot Big Bang.

However, one might argue, it seems we just replaced an initial condition problem with another.

Indeed, for inflation to happen, we need $\varepsilon = -\frac{\dot{H}}{H^2} \ll 1$

In terms of single field dynamics, we need the inflaton ϕ to find itself in a flat region of its potential $V(\phi)$, and it has to be slow-rolling long enough to source sufficient inflation.

If the potential is flat ($m_{Pl} \frac{\partial V}{\partial \phi} \ll V$ & $m_{Pl}^2 \frac{\partial^2 V}{\partial \phi^2} \ll V$)
then it can be shown that slow-roll is an attractor:
for most ϕ_i & $\dot{\phi}_i$, the trajectory will tend towards $\dot{\phi} = -\frac{V_{,\phi}}{3H}$



Therefore, if we accept that the potential is flat, all we need for inflation is to have suitable field value ϕ.

For models $V(\phi) \propto \phi^n$, $n \geq 2$ (and many other models) this is the case if we assume the initial energy density is of order the Planck scale (reasonable since V should emerge from Planck scale physics)

Then $\rho \sim m_{pe}^4 \Rightarrow V(\phi) \sim m_{pe}^4$

For $\frac{m^2 \phi^2}{2}$, this implies $m^2 \phi^2 \sim m_{pe}^4$

for $m^2 \simeq 10^{-10} m_{pe}^2 \Rightarrow \phi \simeq 10^5 m_{pe}$ and much more than 60 e-folds are realized

At such large energy density quantum fluctuations are quite large: $\Delta \phi_{quantum} \simeq \frac{H}{2\pi} \simeq \frac{1}{2\pi}$

This brings us to the idea of

4.7.1 Eternal inflation

At large energy density, quantum fluctuations are large.

A slow-rolling field, on the other hand, classically rolls by $\Delta\phi_{cl} = \frac{\dot{\phi}}{H}$ over a Hubble time.

We have $\Delta \phi_{cl} = \frac{\dot{\phi}}{H} \simeq - \frac{V_{,\phi}}{3H^2}$

If $|\Delta \phi_{cl}| < \Delta_{qu}\phi = \frac{H}{2\pi}$ then quantum fluctuations dominate and a lot of inflation can be realized in this regime!

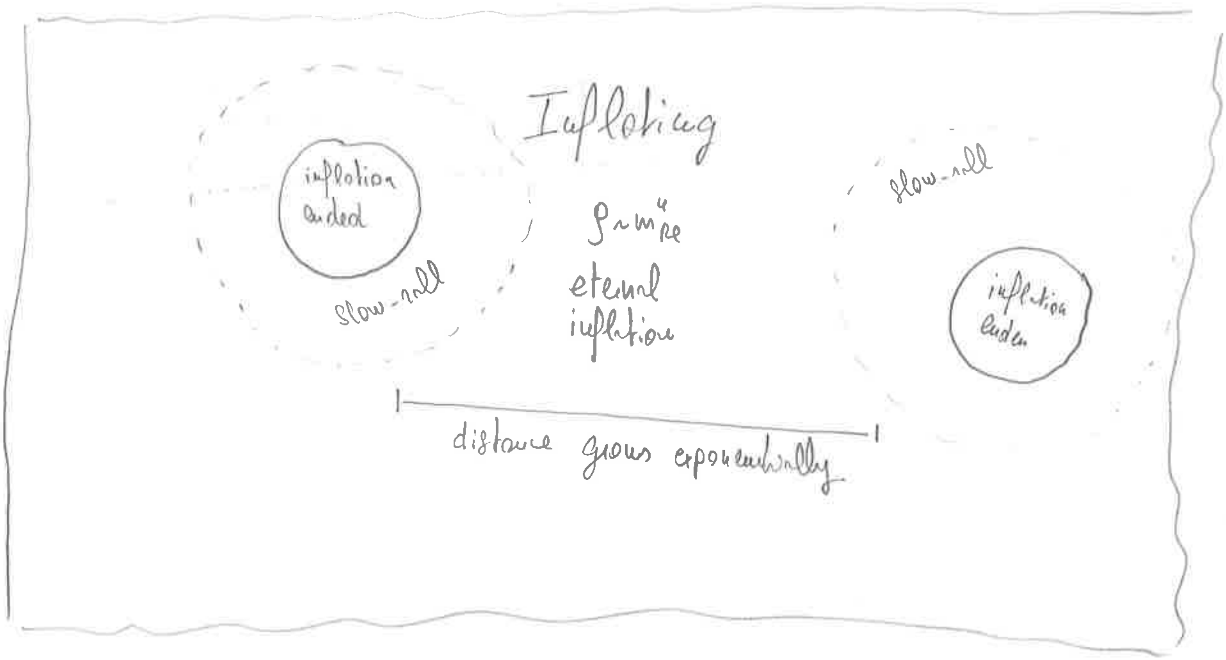
For ϕ^n models, this leads to a picture of the universe dominated by inflating regions.

Indeed, quantum fluctuations can shift the field value to larger or smaller values.

Shift to larger values lead to more expansion, therefore space will be dominated by such regions.

However some regions will exit the eternal inflation regime, and we might live in one of those.

This leads to a view of the universe on the largest scales which is far from homogeneous & isotropic:



4.7.2 Quantum fluctuations at small field values

For ϕ^4 models, QF are suppressed at small field values.

However, if $V \sim V_0 \left[1 + \phi^4 \right]$ quantum fluctuations can dominate at small field values also.

This regime is called quantum diffusion and can be important for understanding the initial conditions in models involving symmetry breaking phase transition

5. After inflation: Reheating the universe

End of inflation: field trajectory oscillates around the global minimum of the potential

For a single field:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

if oscillation frequency $\gg H$, then

$$\ddot{\phi} + V_{,\phi} \simeq 0$$

Multiply by ϕ:

$$\phi\ddot{\phi} + \phi V_{,\phi} = \frac{\partial}{\partial t}(\phi\dot{\phi}) - \dot{\phi}^2 + \phi V_{,\phi} \simeq 0$$

Average of an oscillation period:

$$\frac{1}{t_{osc}} \int_0^{t_{osc}} \frac{\partial}{\partial t}(\dot{\phi}\phi)\, dt = \frac{1}{t_{osc}} \dot{\phi}\phi \Big|_0^{t_{osc}} = 0$$

$$\Rightarrow \quad \langle \dot{\phi}^2 \rangle \simeq \langle \phi V_{,\phi} \rangle$$

$$\Rightarrow \quad \langle \rho_\phi \rangle \simeq \langle V \rangle + \frac{\langle \dot{\phi}^2 \rangle}{2} = \langle V \rangle + \frac{\langle \phi V_{,\phi} \rangle}{2}$$

$$\langle p_\phi \rangle = -\langle V \rangle + \frac{\langle \phi V_{,\phi} \rangle}{2}$$

$$w \simeq \frac{\langle \phi V_{,\phi} \rangle - 2\langle V \rangle}{\langle \phi V_{,\phi} \rangle + 2\langle V \rangle}$$

$$V \propto \phi^n \quad \Rightarrow \quad w \simeq \frac{n-2}{n+2}$$

For $n = 2$ $\quad w = 0 \quad \Rightarrow$ dust

$n = 4$ $\quad w = \frac{1}{3} \quad \Rightarrow$ radiation

In order to restore the Hot Big Bang, energy has to be transfered to Standard Model particles (presumably via some intermediate stages)

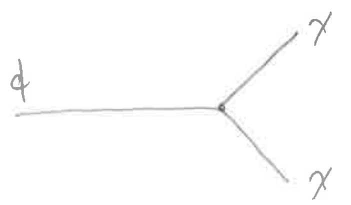
This process is called reheating.

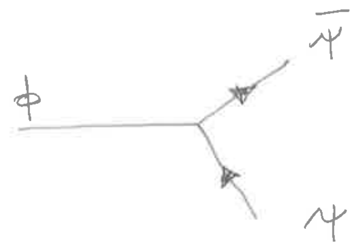
5.1 Perturbative reheating

Consider an inflaton ϕ of mass m_ϕ $\rightarrow V(\phi) = \frac{m_\phi^2 \phi^2}{2}$. At the end of inflation, its coupling to a scalar field χ & a spinor field ψ become relevant

The simplest interactions are 3-legged diagrams:

$$\mathcal{L}_{int} = -g \phi \chi^2 - h \phi \bar{\psi} \psi$$





The resulting decay rates are easily calculated.

For a decay into two final particles, it is

$$\Gamma = \int_0^{\pi} d\cos\theta \int_0^{2\pi} d\phi \; \frac{1}{32\pi^2} |\mathcal{M}|^2 \frac{|P_1|}{m_\phi^2}$$

$$|P_1| = \frac{1}{2m_\phi} \left[\left(m_\phi^2 - (m_1 + m_2)^2 \right) \left(m_\phi^2 - (m_1 - m_2)^2 \right) \right]^{1/2}$$

is momentum in center of mass frame

for $m_1 = m_2 = m$

$$|P_1| = \frac{1}{2} \left(m_\phi^2 - 4m^2 \right)^{1/2} = \frac{m_\phi}{2} \left(1 - \frac{4m^2}{m_\phi} \right)^{1/2}$$

$$\mathcal{M}_{\phi\to\chi\chi} = -ig \Rightarrow |\mathcal{M}_{\phi\to\chi\chi}|^2 = g^2$$

$$\Gamma_{\phi\to\chi\chi} = \frac{2\times 2\pi}{32\pi^2}\, g^2\, \frac{m_\phi}{m^2_\phi}\left(1-\frac{4m^2}{m_\phi^2}\right)^{1/2}$$

$$= \frac{g^2}{8\pi m_\phi}\left(1-\frac{4m^2}{m_\phi^2}\right)^{1/2}$$

$$= \frac{g^2}{8\pi m_\phi} \quad \text{for} \quad m^2 = 0$$

$$\mathcal{M}_{\phi\to\bar\psi\psi} = -ih\, \bar v^{s}(p_1) u^{s'}(p_2) \qquad m_\psi = m_{\bar\psi} = m_f$$

For Dirac spinors

$$\sum_{s,s'} |\mathcal{M}_{\phi\to\bar\psi\psi}|^2 = h^2 \sum_{s,s'} \mathrm{tr}\,(\bar v u)^\dagger \bar v u = h^2\, \mathrm{tr}\,(\not p_1 - m_f)(\not p_2 - m_f)$$

$$= h^2\, \mathrm{tr}\,(\not p_1 \not p_2 - m_f^2) = 4h^2 (p_1 p_2 - m_f^2)$$

$$= 4h^2(\underbrace{E_f^2 + \bar p_f^2}_{\bar p_f^2 = E_f^2 - m_f^2} - m_f^2) = 8h^2(\underbrace{E_f^2}_{\left(\frac{m_\phi}{2}\right)^2} - m_f^2)$$

$$= 2h^2(m_\phi^2 - 4m_f^2)$$

$$= 2h^2 m_\phi^2 \quad \text{for} \quad m_f^2 = 0$$

For Weyl spinor $\sum_{s,s'} |\mathcal{M}_{\phi\to\bar\psi\psi}| = h^2 m_\phi^2$

Consider Weyl spinors

$$\Gamma_{\phi\to\bar\psi\psi} = \frac{h^2 m_\phi}{8\pi}$$

So in our toy model we have

$$\Gamma_{\text{scalar}} = \frac{g^2}{8\pi m_\phi} \quad \& \quad \Gamma_{\text{fermion}} = \frac{h^2 m_\phi}{8\pi}$$

Quantum corrections to the interactions L_{int} can be neglected in the limit $m_\phi > g$ & $m_\phi > h^2$

For $m \ll m_{Pl}$ (remember for $\frac{m_\phi^2 \phi^2}{2}$ A_S constraint gave $m_\phi \sim 10^{-5} m_{Pl}$)

$$\Gamma_{scalar} < \frac{m_\phi}{8\pi} \quad \& \quad \Gamma_{fermion} < \frac{m_\phi^2}{8\pi m_{Pl}}$$ and maximal values s.t

$\Gamma_{scalar} \gg \Gamma_{fermion} \Rightarrow$ inflaton decays mainly into scalar particles

- How fast does the inflaton decay?

The lifetime of a ϕ particle is $\tau_\phi \sim \Gamma_{scalar}^{-1} > m_\phi^{-1} \sim \frac{1}{H}$

$\Rightarrow$ expansion is important

- How does τ_ϕ compare to the oscillating frequency of ϕ?

1st Friedman equation: $m_{Pl}^2 H^2 = \frac{\rho}{3} = \frac{\dot\phi^2}{6} + \frac{m^2\phi^2}{6}$

Using the ansatz $\dot\phi = \sqrt{6 m_{Pl}^2}\, H \sin\theta$

$$m\phi = \sqrt{6 m_{Pl}^2}\, H \cos\theta$$

$$\ddot\phi = \sqrt{6 m_{Pl}^2}\left(\dot H \sin\theta + \dot\theta H \cos\theta\right) \Rightarrow \cancel{\sqrt{6 m_{Pl}^2}}\left(\dot H S_\theta + \dot\theta H C_\theta\right) + 3\cancel{\sqrt{6 m_{Pl}^2}} H^2 S_\theta + \cancel{\sqrt{6 m_{Pl}^2}}\, m H C_\theta = 0$$

$$\left(\dot H S_\theta + \dot\theta H C_\theta\right) + 3H^2 S_\theta + m H C_\theta = 0$$

$$\left(\dot H + 3H^2\right) S_\theta + \left(\dot\theta + m\right) C_\theta = 0$$

one Solution: $\left.\begin{array}{l} \dot H = -3H^2 \\ \dot\theta = -m \end{array}\right\} \Rightarrow \begin{array}{l} H = \frac{1}{3t} \\ \theta = -mt \end{array}$

$$\Rightarrow \phi = \frac{\sqrt{6 m_{Pl}^2}}{3 m_\phi t} \cos(mt) = \Phi(t) \cos(mt)$$

Therefore $\tau_\phi \sim m_\phi^{-1} = \tau_{osc} \Rightarrow$ an inflaton particle typically decays within few oscillations.

The initial number density of ϕ particles can be estimated as $n = \frac{\rho}{m_\phi}$ (# density of non-relativistic particles)

$$\Rightarrow \quad n_\phi = \frac{1}{2m_\phi}\left(\dot{\phi}^2 + m_\phi^2 \phi^2\right) \simeq \frac{6 m_{pl}^2}{2 m_\phi} \frac{1}{9t^2}\left(c_\theta^2 + s_\theta^2\right)$$

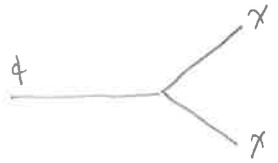
$$= \frac{m_{pl}^2}{3 m_\phi t^2} = \frac{1}{2} \frac{m_\phi \bar{\phi}^2(t)}{m_{pl}^3}$$

At the end of inflation, for $m_\phi \sim 10^{-5} m_{pl}$ & $\bar{\Phi} \sim 1\, m_{pl}$ we have $n_\phi \sim 10^{92}\ \mathrm{cm}^{-3}$!

How does the n_ϕ evolve?

From estimation above we have $n_\phi \propto \frac{1}{t^2} \propto \frac{1}{a^3}$. This is true only if the decay rate $\Gamma_\phi = 0$.

For $\Gamma_\phi \neq 0$ (our case) the evolution of n_ϕ is given by the Boltzmann equation.

For 

we have

$$\frac{1}{a^3} \frac{d(a^3 n_\phi)}{dt} = n_\phi^0 \langle \sigma v \rangle \left(-\frac{n_\phi}{n_{\phi 0}} + \frac{n_\chi^2}{n_\chi^{0\,2}} \right)$$

For $n_\phi \gg n_\chi$, we have

$$\frac{1}{a^3} \frac{d(a^3 n_\phi)}{dt} \simeq -\Gamma_\phi n_\phi$$

where $\Gamma_\phi = \Gamma_{scalars}$

When n_χ becomes larger this is no longer a good description!

We can express $\frac{1}{a^3}\frac{d(n_\phi a^3)}{dt} = -\Gamma_\phi n_\phi$ in terms of ϕ:

$$\frac{1}{a^3}\frac{d(n_\phi a^3)}{dt} = \dot{n}_\phi + 3H n_\phi = -\Gamma_\phi n_\phi$$

$$\Rightarrow \quad \dot{n}_\phi + (3H + \Gamma_\phi) n_\phi = 0$$

$$n_\phi = \frac{1}{2m_\phi}\left(\dot{\phi}^2 + m_\phi^2 \phi^2\right)$$

$$\dot{n}_\phi = \frac{1}{2m_\phi}\left(2\ddot{\phi}\dot{\phi} + 2m_\phi^2 \phi\dot{\phi}\right) = \frac{\dot{\phi}}{m_\phi}\left(\ddot{\phi} + m_\phi^2 \phi\right)$$

$$\Rightarrow \quad \frac{\dot{\phi}}{m_\phi}\left[\ddot{\phi} + m_\phi^2\phi + (3H+\Gamma_\phi)\left(\frac{\dot{\phi}}{2} + \frac{m_\phi^2\phi^2}{\dot{\phi}}\right)\right] = 0$$

during oscillations $\langle \phi V_{,\phi}\rangle = \langle \dot{\phi}^2 \rangle = \langle m_\phi^2 \phi^2 \rangle$

$$\Rightarrow \quad \ddot{\phi} + (3H + \Gamma_\phi)\dot{\phi} + m_\phi^2 \phi = 0$$

Therefore we can interpret the decay rate as a friction term $\Gamma_\phi \dot{\phi}$.

What happens when n_χ becomes larger?

This typically happens soon after the onset of oscillations and is therefore important to take it into account

Lecture 10

5.2 Narrow resonance

We described the decay of inflaton particles ϕ into χ bosons, assuming $m_\phi \gg m_\chi$.
However, this approximation fails soon after the beginning of the reheating process. Bose condensation effects of the χ particles become important and affect the decay rate Γ_ϕ.
We will derive an expression for the corrected decay rate for the case of small coupling $g < m_\phi$ (neglect quantum corrections to interaction terms)
This leads to an effect called narrow resonance.

We start by noting that the oscillating inflaton field can be thought of as a condensate of heavy ϕ particles at rest $(\bar{k} = 0)$

The number density n_φ therefore is

$$n_\varphi = \int \frac{d^3 \bar{k}_\phi}{(2\pi)^3} n_{\bar{k}_\phi} = n_{\bar{k}=0} \qquad n_\chi = \int \frac{d^3 \bar{k}}{(2\pi)^3} n_{\bar{k}}$$

$\hookrightarrow$ occupation number in phase space.

When n_χ is non-zero, in addition to $\phi \to \chi\chi$ we have also the inverse process $\chi\chi \to \phi$ $(\bar{k}_\phi = 0 \Rightarrow \bar{k}_\chi = \pm \bar{k})$
The rates of these processes are proportional to

$$\phi \to \chi\chi : \quad |\langle n_\varphi - 1, n_{\bar{k}} + 1, n_{-\bar{k}} + 1 | \hat{a}^\dagger_{\bar{k}} \hat{a}^\dagger_{-\bar{k}} \hat{a}_\varphi | n_\varphi, n_{\bar{k}}, n_{-\bar{k}} \rangle|^2 = (n_k + 1)(n_{-k} + 1) n_\varphi$$

$$\chi\chi \to \phi : \quad |\langle n_\varphi + 1, n_{\bar{k}} - 1, n_{\bar{k}} + 1 | \hat{a}^\dagger_\varphi \hat{a}_{\bar{k}} \hat{a}_{-\bar{k}} | n_\varphi, n_{\bar{k}}, n_{-\bar{k}} \rangle|^2 = n_k n_{-k} (n_\varphi + 1)$$

Since $n_{\bar{k}} = n_{-\bar{k}} = n_k$ & $n_\varphi \gg 1$

$$\Gamma_\phi \propto n_\varphi (n_k+1)^2 - (n_\varphi+1) n_k^2 = 2n_k n_\varphi + n_\varphi - n_k^2$$
$$= n_\varphi (1+2n_k) - n_k^2$$
$$\simeq n_\varphi (1+2n_k)$$

For $n_k = 0$, we recover $\Gamma_\chi \propto n_\varphi$

$$\Longrightarrow \quad \Gamma_\phi \simeq \Gamma_\chi (1+2n_k)$$

Therefore $n_k \neq 0$ leads to an enhancement of the decay rate (until n_k becomes large & n_φ small).

Let's find an expression for n_k.

The χ particles produced have energy $\frac{m_\phi}{2}$ because of energy conservation.

Their mass of χ depends on the value of the inflaton field ϕ because of the interaction $g\phi\chi^2$

$\Longrightarrow$ mass^2 of χ $= m_\chi^2 + 2g\phi(t)$

The corresponding 3-momentum therefore is

$$k = \left(\left(\frac{m_\phi}{2}\right)^2 - m_\chi^2 - 2g\phi(t) \right)^{1/2}$$

Assume $m_\phi^2 \gg m_\chi^2 + 2g\phi$ & $g\phi \approx g\Phi\cos(m_\phi t)$

This implies that the momenta $\bar{k}$ are contained in a shell of certain width

$$k = \frac{m_\phi}{2}\left(1 - \frac{4m_\chi^2}{m_\phi^2} - \overbrace{\frac{8g\phi(t)}{m_\phi^2}}^{\ll 1}\right)^{1/2} \simeq \frac{m_\phi}{2}\left[\left(1-\frac{4m_\chi^2}{m_\phi^2}\right)^{1/2} + \frac{1}{2m_\phi^2}\frac{-8g\Phi}{\underbrace{\left(1-\frac{4m_\chi^2}{m_\phi^2}\right)^{1/2}}_{\sim 1}}\right]$$

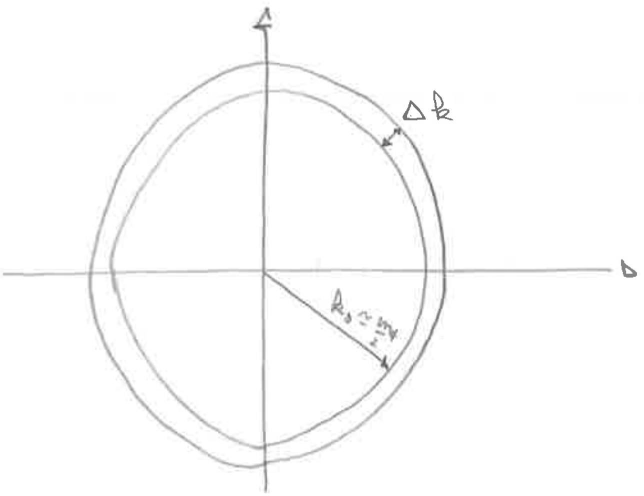
$$\Longrightarrow \quad k - k_0 \approx \frac{m_\phi}{2}\left(-\frac{4g\Phi}{m_\phi^2}\right) \Longrightarrow \Delta k \approx \frac{4g\Phi}{m_\phi} \quad \text{around } k_0$$

$$\Delta k \simeq \frac{4 g \bar{\Phi}}{m_\phi} \ll m_\phi$$

$$n_\chi = \int \frac{d^3 k}{(2\pi)^3} n_k \simeq n_{k_0} \frac{4\pi k_0^2 \Delta k}{(2\pi)^3} \simeq n_{k_0} \frac{2 k_0^2 g \bar{\Phi}}{\pi^2}, \qquad k_0 \simeq \frac{m_\phi}{2} \text{ for } m_\chi \simeq 0$$

$$n_{k_0} = \frac{2\pi^2 n_\chi}{m g \bar{\Phi}}$$

Using $n_\varphi = \frac{1}{2} m \bar{\Phi}^2 \Rightarrow$ $\boxed{n_{k_0} = \frac{\pi^2 \bar{\Phi}}{g} \frac{n_\chi}{n_\varphi}}$



Therefore $\Gamma_\phi \simeq \Gamma_\chi \left(1 + \frac{2\pi^2 \bar{\Phi}}{g} \frac{n_\chi}{n_\phi} \right)$

Bose condensation enhancement

The enhancement becomes important when

$$\frac{2\pi^2 \bar{\Phi}}{g} \frac{n_\chi}{n_\phi} > 1 \Rightarrow n_\chi > \frac{g}{2\pi^2 \bar{\Phi}} n_\varphi$$

At the end of inflation $\bar{\Phi} \sim 1 \Rightarrow$ exceed unity when $\frac{n_\chi}{n_\phi} \sim \frac{g}{2\pi^2}$

The derivation is only valid for small g, $g \ll m_\phi$

& $g\bar{\Phi} << \frac{m_\phi^2}{8}$

For $m_\phi \sim 10^{-5}$ $\Rightarrow$ at most $\frac{n_\chi}{n_\phi} \sim 10^{-10}$ before condensate enhancement becomes important!